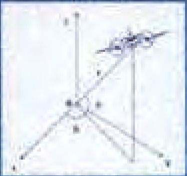
Kalman Filtering Techniques for Radar Tracking



K. V. Ramachandra

Kalman Filtering Techniques for Radar Tracking

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MARCEL DEKKER, INC. NEW YORK • BASEL

ISBN: 0-8247-9322-6

This book is printed on acid-free paper.

Headquarters
Marcel Dekker, Inc.
270 Madison Avenue, New York, NY 10016
tel: 212-696-9000; fax: 212-685-4540

Eastern Hemisphere Distribution
Marcel Dekker AG
Hutgasse 4, Postfach 812, CH-4001 Basel, Switzerland
tel: 41-61-261-8482; fax: 41-61-261-8896

World Wide Web
http://www.dekker.com

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Current printing (last digit):
10 9 8 7 6 5 4 3 2 1

PRINTED IN THE UNITED STATES OF AMERICA

To Bhagawan Sri. Sathya Sai Baba

Preface

The Kalman filter theory published in 1960 significantly boosted the development of sophisticated digital filter algorithms for tracking space vehicles. As a result, a large number of tracking filters have been developed and their algorithms published in journals.

Tracking of objects based on Kalman filter theory has become an established technique of fundamental importance in both engineering applications and scientific investigations. The central problem is that radar and sonar systems, optical telescopes, and infrared sensors used in civil and defense applications require updated information obtained continuously on the parameters that describe the dynamics of such targets as satellites, missiles, aircraft, ships, submarines, RPVs, and other objects having a significant relative motion with respect to the sensor.

Recent developments such as track-while-scan systems, phased array radar tracking, airborne radar tracking, multitarget tracking, multisensor tracking, and multitarget multisensor tracking have not only increased the scope of tracking technology but also added new dimensions to it.

Specifically, the position of a target such as an aircraft or similar vehicle is measured at discrete intervals of time by an automatic track-while-scan radar sensor, and the measurements are reported to a radar data processor (RDP). The reports obtained from successive radar scans are processed by the RDP and suitable tracks are formed.

A computer tracking filter is used to smooth the report data corrupted by range noise and angular noise caused by the electronic and mechanical components of the measuring device.

The tracking filter is the most important component of an RDP/surveillance system. It processes the target radar measurements, reduces the measurement errors, estimates the position, velocity, and/or acceleration of the target at any instant of time, and predicts the future position of the target. Hence the tracking filter is the heart and soul of a radar data processing system.

This book deals with the development of different types of tracking filters based on the Kalman filtering techniques for radar tracking applications.

Chapter 1 presents the discrete-time formulation of Kalman filter, the continuous-time and continuous-discrete-time formulations of Kalman-Bucy filters, and the extended Kalman filter.

Chapter 2 deals with the application of Kalman filter theory for developing one-dimensional trackers for tracking targets such as an aircraft moving with constant velocity or constant acceleration motion when position measurements are obtained by a track-while-scan radar sensor through random noise. Three models are discussed and their steady state results obtained analytically.

Chapter 3 deals with the extension of one-dimensional models to two dimensions for tracking an aircraft or any other space vehicle by a two-dimensional track-while-scan radar that measures the range and bearing of the target. The tracking operation is assumed to be done in the cartesian coordinate system, and the coupling between the quantities measured by the radar and the cartesian coordinate system is explicitly considered in the development of two-dimensional models.

Chapter 4 deals with the extension of one-dimensional models to three dimensions for tracking an aircraft or any other target with range, bearing, and elevation measurements obtained by a three-dimensional track-while-scan radar sensor. The tracking operation is assumed to be performed in cartesian coordinates and the coupling is explicitly considered.

Chapter 5 deals with the continuous-time Kalman tracking filters with position measurements. Fitzgerald's steady state solutions of ECV and ECA models are discussed. The general solution of the second-order ECV model of Nash is given. The random walk velocity model and the random walk acceleration model are also presented.

Chapter 6 deals with the continuous-discrete-time Kalman tracking filters with position measurements. Singer's ECA model and Fitzgerald's steady state performance analysis are discussed. Vaughan's nonrecursive algorithm is briefly described. The steady state results of ECV and ECA filters based on Vaughan's nonrecursive algorithm are presented. Finally, Beuzit's steady state results of the ECA filter obtained by a comparison of Kalman and Wiener filter theories are presented.

Chapter 7 deals with continuous-discrete-time one-dimensional models with position and velocity measurements. A two-state model, an ECV target tracking filter, Fitzgerald's steady state analysis of the ECA model, and a three-state filter are discussed and their steady state solutions are presented.

Chapter 8 deals with continuous-time one-dimensional tracking filters with position and velocity measurements. A two-state model and a three-state model are discussed.

Chapter 9 deals with maneuvering target tracking filters. Bar-Shalom–Birmiwal's model is discussed and Blom–Bar-Shalom's interacting multiple model is presented.

Chapter 10 deals with tracking a maneuvering target in clutter. Validation region or gate, the probabilistic data association filter, and Bar-Shalom–Chang–Blom's model for automatic track formation are discussed.

Chapter 11 deals with an introduction to multitarget tracking. The JPDAF and Reid's algorithm are mentioned.

This book provides enough information in the selection of trackers to meet the requirements of practicing engineers. It also provides sufficient material for advanced students to take up further work in the field.

K. V. Ramachandra

Acknowledgments

I wish to express my gratitude to Dr. A. P. J. Abdul Kalam, Scientific Adviser to the Minister for Defence, and Dr. V. K. Aatre, Chief Controller of Research & Development, DRDO, New Delhi, and Dr. G. M. Cleetus, Director, Mr. N. P. Ramasubba Rao, former Director, Mr. K. U. Limaye, Associate Director, Dr. S. Christopher, Divisional Officer of "C" Radar Division, Mr. K. N. Dinesh Kumar, Scientist "D," Mr. J. Paramashivan, Technical Officer "B," and other colleagues at the Electronics and Radar Development Establishment, Bangalore, India, for their help and encouragement in the development of the book. Credit goes to Miss S. Sukanya for the cover artwork.

I am also grateful to Mr. R. P. Mohan of Bharath Electronics Ltd., Bangalore; Mr. B. N. Ramesh of Metabyte, Fremont, California; Mr. B. R. Mohan and Mrs. B. R. Geetha of National College, Bangalore; and Mrs. B. R. Gayathri of Fremont, California, for their invaluable interest and help in the development of the book.

I am deeply indebted to Mrs. Chaya Ramachandra for her patience and perseverance during the preparation of the book.

Finally, I wish to thank Dr. Y. Bar-Shalom, Distinguished Professor, University of Connecticut, Storrs, Connecticut, for his help in the development of the book.

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1.1 INTRODUCTION

The Kalman filter has made a dramatic impact on linear estimation because of its adaptability for implementation on a digital computer for on-line estimation and usefulness of the state-space approach. Today the Kalman filter is an established technique widely applied in the fields of navigation, guidance, attitude control, satellite orbit determination, aircraft and missiles tracking, radar, sonar and biomedical signal processing, reentry of space vehicles, etc. [1-11]. Many new applications of this powerful technique are being reported in various fields of engineering and technology.

The general-discrete time formulation of the Kalman filter [1], the continuous-time Kalman-Bucy filter [2], and the continuous-discrete-time Kalman-Bucy filter [2, 6] are presented in this chapter.

1.2 DISCRETE-TIME KALMAN FILTER

The statistical model of the signal process is assumed to be described by the discrete, linear, vector matrix equation of the form [1–11]

$$X_{k+1} = F_k X_k + G_k W_k \tag{1.1}$$

where

X_k = n-dimensional state vector at the kth stage
F_k = $n \times n$ transition matrix
G_k = $n \times r$ input distribution matrix
W_k = r-dimensional random input vector

W_k is assumed to be white gaussian with the following properties:

$$E\{W_k\} = 0 \tag{1.2}$$
$$E\{W_j W_k^T\} = Q\delta_{jk}$$

where Q is the $r \times r$ covariance matrix of the process noise W_k and δ_{jk} is the Dirac delta function.

The statistical model of the measurement process is described by

$$Z_k = H_k X_k + V_k \tag{1.3}$$

where Z_k is the m-dimensional measurement vector, H_k is the $m \times n$ observation matrix, and V_k is the m-dimensional random disturbance vector that is corrupting the measurements.

V_k is assumed to be white gaussian with the following properties:

$$E\{V_k\} = 0 \tag{1.4}$$
$$E\{V_j V_k^T\} = R\delta_{jk}$$

where R is the $m \times m$ covariance matrix of the measurement noise V_k.

The random sequences W_k and V_k are assumed to be independent of each other and also independent of the initial state X_0 with the following properties:

$$E\{X_0\} = 0 \tag{1.5}$$
$$E\{W_j V_k^T\} = 0$$
$$E\{X_0 W_k^T\} = 0$$
$$E\{X_0 V_k^T\} = 0$$

Now an estimate of the state vector X_k, based upon the knowledge of the measurements in Z_j, where

$$Z_j \triangleq \{Z_1, Z_2, \ldots, Z_j\} \tag{1.6}$$

is denoted as $\hat{X}_{k|j}$. Specifically, $k > j$ denotes a predicted estimate, $k < j$ denotes a smoothed estimate, and $k = j$ denotes a filtered estimate.

If the mean square error is chosen as the optimal criterion, then Kalman [1] has shown that the minimizing estimate is given by

$$\hat{X}_{k|j} = E\{X_k | Z_j\} \tag{1.7}$$

where $E\{X_k | Z_j\}$ denotes the conditional expectation of X_k given the knowledge of Z_j.

A complete knowledge of the statistical model constitutes the knowledge of F_k, H_k, G_k, Q, R and the structure defined in Eqs. (1.2), (1.4), and (1.5). If this is true, then the mean square error filtered estimate, $\hat{X}_k$, is given by the Kalman filter algorithm as

$$\hat{X}_k = \tilde{X}_k + K_k(Z_k - H_k \tilde{X}_k) \tag{1.8}$$

with

$$\tilde{X}_k = F_{k-1} \hat{X}_{k-1} \tag{1.9}$$

where $\tilde{X}_k$ is the optimum estimate of the state vector before processing the measurement Z_k and K_k is the Kalman gain matrix given by

$$K_k = \tilde{P}_k H_k^T (H_k \tilde{P}_k H_k^T + R)^{-1} \tag{1.10}$$

where $\tilde{P}_k$ is the covariance matrix of estimation errors before processing the measurement Z_k and is computed recursively as

$$\tilde{P}_{k+1} = F_k \hat{P}_k F_k^T + G_k Q G_k^T \tag{1.11}$$

$\hat{P}_k$ is the covariance matrix of estimation errors after processing the observation and is given by

$$\hat{P}_k = (I - K_k H_k) \tilde{P}_k \tag{1.12}$$

Equations (1.11) and (1.12) are referred to as the discrete Riccati equations. $\hat{P}_k$ may also be expressed equivalently as

$$\hat{P}_k = (I + \tilde{P}_k H_k^T R^{-1} H_k)^{-1} \tilde{P}_k \tag{1.13}$$

or

$$\hat{P}_k^{-1} = \tilde{P}_k^{-1} + H_k^T R^{-1} H_k \tag{1.14}$$

The Kalman gain matrix K_k given by (1.10) may also be expressed in terms of $\hat{P}_k$ as

$$K_k = \hat{P}_k H_k^T R^{-1} \tag{1.15}$$

If the gaussian assumption is dropped, then the Kalman filter is the minimum mean square error linear filter.

From Eqs. (1.8) to (1.12), it can be seen that the Kalman filter is a recursive estimator so that it processes the measurements as they are generated in real time without any growing memory problem. Thus it is easy to implement on the digital computer for on-line estimation.

As Fitzgerald puts it in [11], the advantages of the Kalman filter are as follows:

1. The steady state restriction is removed so that optimum results are achieved even during start-up transients.
2. Systems dynamics and noise characteristics may be nonstationary.
3. Both continuous- and discrete-time formulations are possible.
4. Measurements may be treated whenever they become available (not necessarily at a constant rate) and may consist of any function of the state variables.
5. Large number of state variables may be handled in a straightforward way (although with increased computational cost).
6. A by-product of the filter computations is the generation of a covariance matrix which provides a statistical measure of performance in the form of variances and covariances of the estimation errors.

The fundamental work of Kalman in linear filtering theory has been followed by a large number of papers and reports on the subject discussing its applications in various fields of engineering and technology.

The application of Kalman filter theory requires the definition of a linear mathematical model describing the system for which the application is intended.

1.3 CONTINUOUS-TIME KALMAN-BUCY FILTER

The dynamic model for the continuous-time case is described by a vector first-order differential equation of the form [2–4]

$$\dot{x} = Fx + Gu \tag{1.16}$$

where x is an n-dimensional state vector and $\dot{x}$ is its time derivative. F is an $n \times n$ matrix, G is an $n \times r$ matrix, and u is an r-dimensional white noise vector with covariance

$$E[u(t)u^T(\tau)] = Q\delta(t-\tau)$$

where $\delta(t-\tau)$ is a Dirac delta function. The output of the measurement system is given by

$$z = Hx + v \tag{1.17}$$

z is an m-dimensional vector and v is an m-dimensional white noise vector with covariance given by

$$E[v(t)v^T(\tau)] = R\delta(t-\tau)$$

The problem is to find the best estimate in the mean square sense of $x(t)$, $\hat{x}(t|t)$, given $z(\tau)$ for $0 \leq \tau \leq t$. The optimal filter is a linear dynamical system of the form

$$\dot{\hat{x}}(t|t) = F\hat{x}(t|t) + K(t)[z - H\hat{x}(t|t)] \tag{1.18}$$

whose initial state is x_0, and where

$$K(t) = P(t)H^T R^{-1} \tag{1.19}$$

$P(t)$ is the covariance matrix of the optimal error

$$P(t) = E\{[x(t) - \hat{x}(t|t)][x(t) - \hat{x}(t|t)]^T\} \tag{1.20}$$

This covariance matrix is given by a solution to the matrix Riccati equation

$$\dot{P} = FP + PF^T - PH^T R^{-1} HP + GQG^T \tag{1.21}$$

1.4 CONTINUOUS-DISCRETE-TIME KALMAN-BUCY FILTER

The linear dynamical system is described by the differential equation [2, 3] as

$$dx_t = F(t)x_t\, dt + G(t)\, d\beta_t \qquad t \geq t_0 \tag{1.22}$$

where x_t is an n-dimensional state vector, F and G are $n \times n$ and $n \times r$ continuous-matrix-time functions, and β_t is an r-dimensional process noise vector with covariance

$$E\{d\beta_t\, d\beta_t^T\} = Q(t)\, dt$$

The discrete linear observations are taken at time instants t_k and the measurement equation is described by

$$z_k = H_k x_k + v_k \qquad t_{k+1} > t_k \geq t_0 \tag{1.23}$$

where z_k is an m-dimensional observation vector, H is an $m \times n$ observation matrix, and v_k is an m-dimensional vector white Gaussian sequence with zero mean and covariance R_k.

Integrating (1.22) over the interval $[t_k, t_{k+1}]$, we get

$$x_{t_{k+1}} = F(t_{k+1}, t_k)x_{t_k} + \int_{t_k}^{t_{k+1}} G(\tau)\, d\beta_\tau \tag{1.24}$$

where F is a state transition matrix of (1.22). Equation (1.24) may be written as

$$x_{k+1} = Fx_k + w_{k+1} \tag{1.25}$$

where

$$w_{k+1} = \int_{t_k}^{t_{k+1}} G(\tau)\, d\beta_\tau$$

w_k is a zero mean white gaussian sequence with covariance Q_k. Thus the continuous-discrete-time filter is expressed as a discrete filter and the properties of the continuous-discrete-time filter are the same as the discrete filter [3].

1.5 SUMMARY

The discrete-time formulation of the Kalman filter is presented in Section 1.2. The continuous-time Kalman-Bucy filter is given in Section 1.3. The continuous-discrete Kalman filter is discussed in Section 1.4. The details of derivation of these filters are omitted here and are available in Refs. 1 to 6.

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Discrete-Time One-Dimensional Tracking Filters

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2.1 INTRODUCTION

The advent of Kalman filter theory provided significant impetus to the development of sophisticated digital filter algorithms for tracking space vehicles making use of the noisy measurements obtained by a track-while-scan radar sensor.

A number of Kalman filter-based algorithms are available for performing the aircraft-tracking operation either in the cartesian coordinate system or in the polar coordinate system or other coordinate systems. In each system of tracking operation, there are algorithms dealing with aircraft moving with constant velocity perturbed by a zero mean random acceleration or those moving with constant acceleration perturbed by a zero mean plant noise which accounts for maneuvers and/or other random factors, etc. In each of these cases, algorithms may be further classified as dealing with one-, two- or three-dimensional models.

Turns, evasive maneuvers, acceleration due to atmospheric turbulence, etc. are all regarded as perturbations upon the aircraft trajectory.

The application of Kalman filtering techniques for the development of one-dimensional trackers for estimating position, velocity, and acceleration of a space vehicle is illustrated and three models are presented in this chapter.

2.2 A TWO-STATE FILTER: FRIEDLAND'S MODEL*

Consider an aircraft or similar space vehicle moving with constant velocity perturbed by a zero mean random acceleration. The position of the vehicle is assumed to be measured by a track-while-scan radar sensor at uniform sampling intervals of time T seconds and all measurements are noisy. The problem is to obtain the optimum estimates of position and velocity of the vehicle.

This model, developed by Friedland [1], assumes that each component of the vehicle position is independently measured by a radar sensor in the cartesian coordinate system with constant accuracy, and that the observation errors have zero mean and are uncorrelated.

2.2.1 Dynamic Model

As the model assumes that each position coordinate is measured independently, each coordinate is uncoupled from the other two and hence can be treated separately. For each coordinate, the vehicle dynamics is assumed to be described by

$$\begin{aligned} x_{n+1} &= x_n + \dot{x}_n T + \tfrac{1}{2} a_n T^2 \\ \dot{x}_{n+1} &= \dot{x}_n + a_n T \end{aligned} \tag{2.1}$$

* © Kearfott Guidance and Navigation Corporation, Wayne, New Jersey.

where x_n = vehicle position at scan n
$\dot{x}_n$ = vehicle velocity at scan n
a_n = acceleration acting on the vehicle at scan n
T = interval between observations

In the model given by (2.1), the acceleration is assumed to be a random constant between successive observations with zero mean and uncorrelated with its values at other intervals, i.e.,

$$E\{a_n\} = 0 \tag{2.2}$$
$$E\{a_n^2\} = \sigma_a^2 = \text{constant} \qquad \text{for all } n$$
$$E\{a_n a_k\} = 0 \qquad \text{for } n \neq k$$

In the vector-matrix form, the vehicle dynamics (2.1) may be written as

$$X_{n+1} = FX_n + Ga_n \tag{2.3}$$

with

$$X_n = \begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix} \tag{2.4}$$

$$F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{2.5}$$

and

$$G = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \tag{2.6}$$

X_n is the vehicle state vector consisting of position and velocity components, F is the transition matrix, and G is the input distribution matrix.

2.2.2 Measurement Model

The position of the vehicle is assumed to be measured by a radar at uniform intervals of time T seconds and each observation is noisy. The measurement equation is given by

$$x_m(n) = x_n + v_n \tag{2.7}$$

where

$x_m(n)$ = measured position at scan n
x_n = the true position at scan n
v_n = random noise corrupting the measurement at scan n

The statistical properties of the noise are assumed to be

$$E\{v_n\} = 0 \tag{2.8}$$

$$E\{v_n^2\} = \sigma_x^2 = \text{constant} \qquad \text{for all } n$$

$$E\{v_n v_k\} = 0 \qquad \text{for } n \neq k$$

In terms of the state vector X_n, (2.7) may be written as

$$x_m(n) = HX_n + v_n \tag{2.9}$$

with

$$H = [1 \quad 0] \tag{2.10}$$

2.2.3 Filtering Equations

Now (2.3) and (2.9) are in the standard format for application of the Kalman filter theory. Hence from (1.8) and (1.9), the optimum estimate of the state vector is given by

$$\hat{X}_n = \tilde{X}_n + K_n[x_m(n) - H\tilde{X}_n] \tag{2.11}$$

with

$$\tilde{X}_n = F\hat{X}_{n-1} \tag{2.12}$$

where

$$\hat{X}_n = \begin{bmatrix} \hat{x}_n \\ \hat{\dot{x}}_n \end{bmatrix}$$

is the optimum estimate of the state vector after the measurement $x_m(n)$ is processed, and

$$\tilde{X}_n = \begin{bmatrix} \tilde{x}_n \\ \tilde{\dot{x}}_n \end{bmatrix}$$

is the optimum estimate of the state vector before the measurement $x_m(n)$ is processed.

The Kalman gain matrix K_n is given by

$$K_n = \tilde{P}_n H^T (H\tilde{P}_n H^T + R)^{-1} \tag{2.13}$$

where $R = \sigma_x^2$ is the variance of the measurement noise and $\tilde{p}_n$ is the covariance matrix of estimation errors before processing the measurement $x_m(n)$ computed recursively using the variance equation (1.11) as

$$\tilde{P}_{n+1} = F\hat{P}_n F^T + GQG^T \tag{2.14}$$

where $Q = \sigma_a^2$ is the variance of random acceleration and $\hat{P}_n$ is the covariance matrix of estimation errors after processing the measurement $x_m(n)$. From (1.12), $\hat{P}_n$ is given by

$$\hat{P}_n = (I - K_n H)\tilde{P}_n \tag{2.15}$$

2.2.4 Steady State Analysis

In the steady state ($n \to \infty$),

$$\begin{aligned} \tilde{P}_{n+1} &= \tilde{P}_n = \tilde{P} \quad \text{(say)} \\ \hat{P}_{n+1} &= \hat{P}_n = \hat{P} \\ K_{n+1} &= K_n = K \end{aligned} \tag{2.16}$$

Hence, in the steady state, Eqs. (2.13) to (2.15) may be written as:

$$K = \tilde{P}H^T(H\tilde{P}H^T + R)^{-1} \tag{2.17}$$

$$\tilde{P} = F\hat{P}F^T + GQG^T \tag{2.18}$$

$$\hat{P} = (I - KH)\tilde{P} \tag{2.19}$$

Equations (2.18) and (2.19) may be combined into a single equation as

$$\tilde{P} - GQG^T = F(I - KH)\tilde{P}F^T \tag{2.20}$$

2.2.5 Steady State $\tilde{P}$ Matrix

If the covariance matrix $\tilde{P}$ is defined as

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{12} & \tilde{P}_{22} \end{bmatrix} \tag{2.21}$$

then the normalized covariances may be expressed as:

$$\begin{aligned} \tilde{Y}_{11} &= \tilde{P}_{11}/\sigma_x^2 \\ \tilde{Y}_{12} &= \tilde{P}_{12}/(\sigma_x \sigma_a T) \\ \tilde{Y}_{22} &= \tilde{P}_{22}/(\sigma_a^2 T^2) \end{aligned} \tag{2.22}$$

Now evaluating (2.20) gives rise to the following three nonlinear equations:

$$4(1+\tilde{Y}_{11})(2r\tilde{Y}_{12}+4\tilde{Y}_{22}+1)=(r\tilde{Y}_{11}+4\tilde{Y}_{12})^2 \tag{2.23}$$

$$2(1+\tilde{Y}_{11})(2\tilde{Y}_{22}+1)=\tilde{Y}_{12}(r\tilde{Y}_{11}+4\tilde{Y}_{12}) \tag{2.24}$$

$$(1+\tilde{Y}_{11})=\tilde{Y}_{12}^2 \tag{2.25}$$

where

$$r=4\sigma_x/(\sigma_a T^2) \tag{2.26}$$

The ratio r is a dimensionless parameter. Friedland [1] has termed this as a sort of noise-to-signal ratio since σ_x is the sensor standard deviation (ft) and $\sigma_a T^2/2$ is the position error due to a constant acceleration of σ_a (ft/s^2).

The solution to the three nonlinear equations (2.23) to (2.25) is given separately in Appendix 2A. After considerable algebraic manipulations, the steady state predicted covariances may be found as:

$$\begin{aligned}\tilde{Y}_{11}&=d(d+1)^2/r^2\\ \tilde{Y}_{12}&=(d+1)^2/2r\\ \tilde{Y}_{22}&=(d+1)/2\end{aligned} \tag{2.27}$$

where

$$d=\sqrt{1+2r}$$

2.2.6 Steady State Gain Vector

In the steady state, the gain vector is a constant. Let K be defined as

$$K=\begin{bmatrix}K_1\\ K_2\end{bmatrix} \tag{2.28}$$

Putting H and $\tilde{P}$ from (2.10) and (2.21) in (2.17) and simplifying using (2.25), we get

$$\begin{aligned}K_1&=\tilde{Y}_{11}/\tilde{Y}_{12}^2\\ K_2&=4/rT\tilde{Y}_{12}\end{aligned} \tag{2.29}$$

Using (2.27), (2.29) may be written as

$$\begin{aligned}K_1&=d(d-1)^2/r^2\\ K_2&=2(d-1)^2/Tr^2\end{aligned} \tag{2.30}$$

If the normalized gains are defined as

$$G_1 = K_1 \qquad (2.31)$$
$$G_2 = TK_2$$

then using (2.30) in (2.31), we get

$$G_1 = d(d-1)^2/r^2 \qquad (2.32)$$
$$G_2 = 2(d-1)^2/r^2$$

2.2.7 Steady State $\hat{P}$ Matrix

If the filtered covariance matrix is defined as

$$\hat{P} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12} & \hat{P}_{22} \end{bmatrix} \qquad (2.33)$$

then the normalized elements of the $\hat{P}$ matrix may be written as:

$$\hat{Y}_{11} = \hat{P}_{11}/\sigma_x^2 \qquad (2.34)$$
$$\hat{Y}_{12} = \hat{P}_{12}/(\sigma_x \sigma_a T)$$
$$\hat{Y}_{22} = \hat{P}_{22}/(\sigma_a^2 T^2)$$

(2.33) and (2.34) are the same as those defined in (2.21) and (2.22) except that the tildes are replaced by hats. Using (2.21) and (2.28) in (2.19), the $\hat{P}$ matrix may be written as

$$\hat{P} = \begin{bmatrix} (1-K_1)\tilde{P}_{11} & (1-K_1)\tilde{P}_{12} \\ \tilde{P}_{12} - K_2\tilde{P}_{11} & \tilde{P}_{22} - K_2\tilde{P}_{12} \end{bmatrix} \qquad (2.35)$$

Using (2.29), (2.22), and (2.25) in (2.35), the normalized elements of the $\hat{P}$ matrix may be derived as

$$\hat{Y}_{11} = \tilde{Y}_{11}/\tilde{Y}_{12}^2 \qquad (2.36)$$
$$\hat{Y}_{12} = 1/\tilde{Y}_{12}$$
$$\hat{Y}_{22} = \tilde{Y}_{22} - 1$$

Using (2.27) in (2.36), we get, after simplification,

$$\hat{Y}_{11} = d(d-1)^2/r^2 \qquad (2.37)$$
$$\hat{Y}_{12} = (d-1)^2/2r$$
$$\hat{Y}_{22} = (d-1)/2$$

2.2.8 Numerical Results

From (2.27), (2.32), and (2.37), it is seen that all the steady state normalized covariances and gains are functions of the dimensionless parameter r. As an example, for

$$r = 10.0 \text{ (say)}$$

the steady state $\tilde{Y}$, $\hat{Y}$, and G matrices can be evaluated from Eqs. (2.27), (2.37), and (2.32) and the results may be found as given below.

$$\tilde{Y} = \begin{bmatrix} 1.4282 & 1.5583 \\ 1.5583 & 2.7913 \end{bmatrix}$$

$$\hat{Y} = \begin{bmatrix} 0.5882 & 0.6417 \\ 0.6417 & 1.7913 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.5882 \\ 0.0717 \end{bmatrix}$$

The graphs of $\tilde{Y}_{11}$ and $\hat{Y}_{11}$ are plotted against r in Figure 2.1. The velocity accuracy before and after position determination is plotted against r in Figure 2.2, and the normalized velocity gain is plotted against r in Figure 2.3.

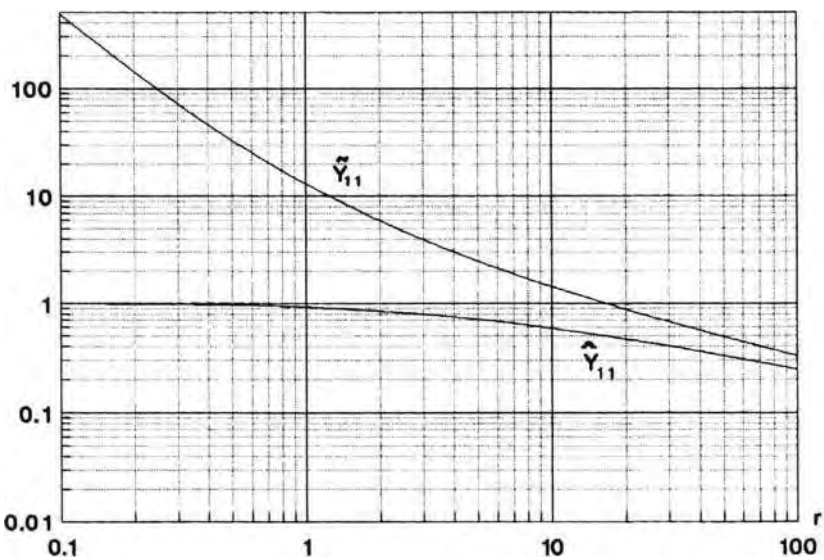

Figure 2.1 Position accuracy before and after measurements. (From Ref. 1; © 1973—IEEE.)

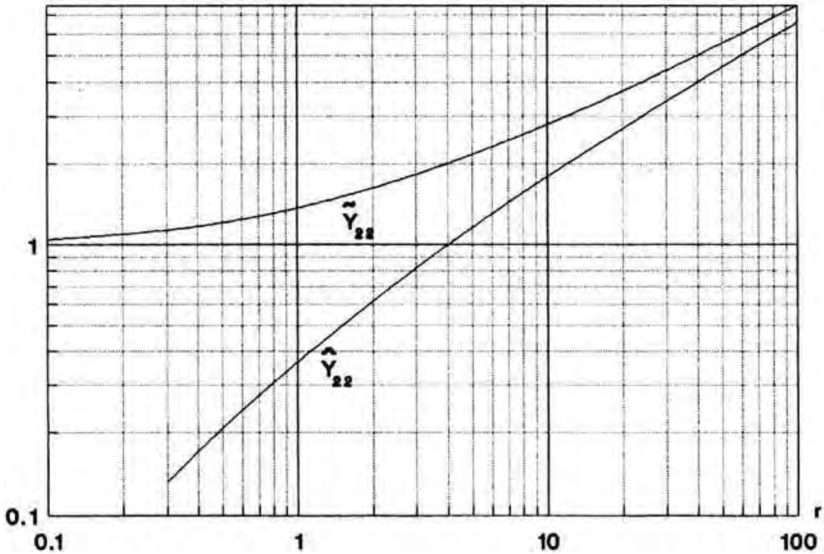

Figure 2.2 Velocity accuracy before and after measurements. (From Ref. 1; © 1973—IEEE.)

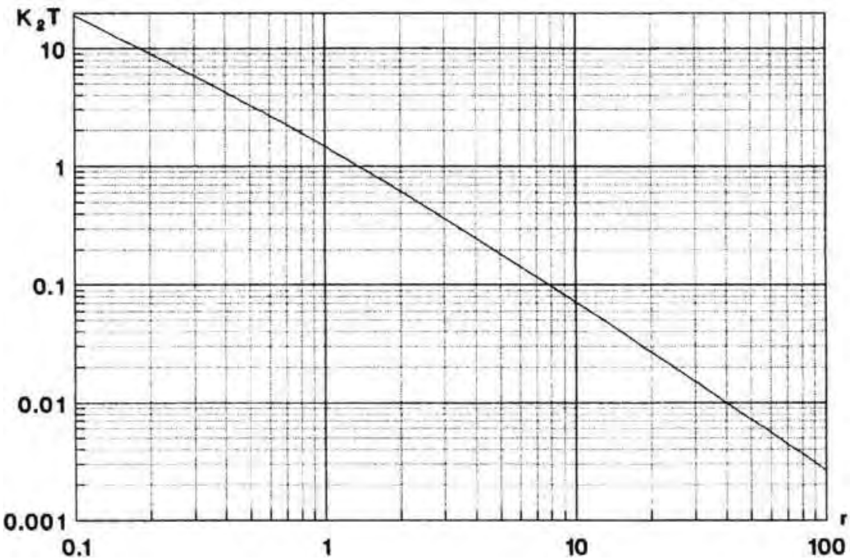

Figure 2.3 Normalized velocity gain as a function of r. (From Ref. 1; © 1973—IEEE.)

The unnormalized covariances and gains may be evaluated for known values of σ_a or σ_x and T. Let

$$T = 4 \text{ s}$$

$$\sigma_a = 0.01414 \text{ nm/s}^2$$

then from (2.26), for $r = 10$, we get

$$\sigma_x = 0.5656 \text{ nm}$$

As $\tilde{Y}$, $\hat{Y}$, and G matrices are known for these parameters, $\tilde{P}$, $\hat{P}$, and K matrices may be evaluated from (2.22), (2.34), and (2.31) as

$$\tilde{P} = \begin{bmatrix} 0.4569 & 0.0498 \\ 0.0498 & 0.0089 \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} 0.1882 & 0.0205 \\ 0.0205 & 0.0057 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.5882 \\ 0.2866 \end{bmatrix}$$

If $\tilde{P}$, $\hat{P}$, and K matrices are evaluated by executing the Kalman filter matrix equations (2.17) to (2.19) recursively to the steady state, we get the same results as given above.

2.2.9 Interpretation of Steady State Results

From (2.27) and (2.37), it is seen that the steady state covariance matrices before and after processing a measurement are not the same: $\tilde{P} \neq \hat{P}$. In fact, $\hat{P} < \tilde{P}$. This means that errors are reduced due to the processing of an observation. However, during the intervals between two observations, the errors increase in accordance with (2.18) due to random acceleration. The steady state is attained when the decrease in error obtained by each observation is exactly equal to the increase in error between observations [1].

2.2.10 Mean Square Values of the Ripple

If the estimates of position and velocity are to be coupled to an automatic control system (e.g., for automatic landing), a high-frequency ripple is produced due to the jumps that occur each time the position and velocity are updated using (2.11). The covariance matrix of this ripple is given by [1]

$$E[(\tilde{x}_n - \hat{x}_n)(\tilde{x}_n - \hat{x}_n)^T] = \tilde{P}_n - \hat{P}_n \tag{2.38}$$

From (2.27) and (2.37), the mean square values of the ripple in position and velocity are given by

$$\tilde{Y}_{11} - \hat{Y}_{11} = 4d_1^2/r^2 \tag{2.39}$$
$$\tilde{Y}_{22} - \hat{Y}_{22} = 1$$

2.2.11 Design Formula for Suitable Sampling Time

For a high level of random acceleration, the past estimates of position and velocity are not of much value in reducing position errors significantly below the level of sensor accuracy. For a very low level of random acceleration, the estimates of position and velocity are highly correlated with previous estimates. Hence the vehicle position can be estimated with greater accuracy than is inherent in the sensor [1]. From Figure 2.1, it may be seen that $\hat{Y}_{11}$ is less than unity, whereas $\tilde{Y}_{11}$ can be greater or smaller than unity and the crossover point occurs for

$$r \approx 16.6 \tag{2.40}$$

Hence from (2.26),

$$\sigma_a T^2 = 0.24\sigma_x \tag{2.41}$$

From (2.41), it is seen that for sufficiently small sampling interval T and/or perturbing acceleration, the position error can be kept below the sensor error even just before observations are made, when the error is highest. This may be realised by using a sampling time T given by

$$T < 0.49\sqrt{\sigma_x/\sigma_a} \tag{2.42}$$

Equation (2.42) is obtained from (2.41) and represents a reasonable design formula for determining a suitable sampling time [1].

2.3 A THREE-STATE FILTER: RAMACHANDRA'S MODEL I

Ramachandra's model I [2] is a one-dimensional model for estimating the optimum steady state position, velocity, and acceleration of an aircraft or similar vehicle moving with a constant acceleration perturbed by a zero mean plant noise which accounts for maneuvers and/or other random factors. The position coordinate of the vehicle is assumed to be measured by radar at uniform intervals of time T seconds through random noise.

2.3.1 Dynamic Model

Each position coordinate of the vehicle is assumed to be described by the following equations of motion:

$$\begin{aligned} x_{n+1} &= x_n + \dot{x}_n T + \ddot{x}_n T^2/2 \\ \dot{x}_{n+1} &= \dot{x}_n + \ddot{x}_n T \\ \ddot{x}_{n+1} &= \ddot{x}_n + a_n \end{aligned} \tag{2.43}$$

where

x_n = vehicle position at scan n
$\dot{x}_n$ = vehicle velocity at scan n
$\ddot{x}_n$ = vehicle acceleration at scan n
a_n = plant noise that perturbs the acceleration of the vehicle and accounts for both maneuvers and other modeling errors
T = sampling time

a_n is assumed to be zero mean and of constant variance σ_a^2 and also uncorrelated with its values at other inrevals; i.e., a_n satisfies the statistical properties given by (2.2).

In vector-matrix form, Eqs. (2.43) can be written as

$$X_{n+1} = FX_n + A_n \tag{2.44}$$

where

$$X_n = \begin{bmatrix} x_n \\ \dot{x}_n \\ \ddot{x}_n \end{bmatrix} \tag{2.45}$$

$$F = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \tag{2.46}$$

$$A_n = \begin{bmatrix} 0 \\ 0 \\ a_n \end{bmatrix} \tag{2.47}$$

2.3.2 Measurement Equation

The measurement equation may be written as

$$x_m(n) = HX_n + v_n \tag{2.48}$$

where $x_m(n)$ is the measured position at scan n, v_n is the random noise corrupting the measurement at scan n, and

$$H = [1 \quad 0 \quad 0] \tag{2.49}$$

The statistical properties of the measurement noise are assumed to be the same as in (2.8).

2.3.3 Filtering Equations

Now Eqs. (2.44) and (2.48) are in the standard form for application of Kalman filtering algorithm. The optimal estimate of the state vector after the measurement is processed is given by

$$\hat{X}_n = \tilde{X}_n + K_n[x_m(n) - H\tilde{X}_n] \tag{2.50}$$

and the state vector $\tilde{X}_n$ before the measurement is given by

$$\tilde{X}_n = F\hat{X}_{n-1} \tag{2.51}$$

The Kalman gain matrix is given by

$$K_n = \tilde{P}_n H^T (H\tilde{P}_n H^T + R)^{-1} \tag{2.52}$$

The predicted covariance matrix $\tilde{P}_n$ is given by

$$\tilde{P}_{n+1} = F\hat{P}_n F^T + Q \tag{2.53}$$

and the filtered covariance $\hat{P}_n$ is given by

$$\hat{P}_n = (I - K_n H)\tilde{P}_n \tag{2.54}$$

Q is the covariance matrix of the plant noise and is given by

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_a^2 \end{bmatrix} \tag{2.55}$$

2.3.4 Steady State Analysis

In the steady state ($n \to \infty$), the relations (2.16) hold good and the relations (2.52) to (2.54) become

$$K = \tilde{P}H^T(H\tilde{P}H^T + R)^{-1} \tag{2.56}$$

$$\tilde{P} = F\hat{P}F^T + Q \tag{2.57}$$

$$\hat{P} = (I - KH)\tilde{P} \tag{2.58}$$

Combining (2.57) and (2.58) in the steady state, we get

$$\tilde{P} - Q = F(I - KH)\tilde{P}F^T \tag{2.59}$$

2.3.5 Steady State $\tilde{P}$ Matrix

If the predicted covariance matrix $\tilde{P}$ is defined as the (3×3) symmetrix matrix given by

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} & \tilde{P}_{13} \\ \tilde{P}_{12} & \tilde{P}_{22} & \tilde{P}_{23} \\ \tilde{P}_{13} & \tilde{P}_{23} & \tilde{P}_{33} \end{bmatrix} \tag{2.60}$$

then the normalized covariances may be written as:

$$\begin{aligned} \tilde{Y}_{11} &= \tilde{P}_{11}/\sigma_x^2 \\ \tilde{Y}_{12} &= \tilde{P}_{12}/(\sigma_x\sigma_a T) \\ \tilde{Y}_{13} &= \tilde{P}_{13}/(\sigma_x\sigma_a) \\ \tilde{Y}_{22} &= \tilde{P}_{22}/(\sigma_a^2 T^2) \\ \tilde{Y}_{23} &= \tilde{P}_{23}/(\sigma_a^2 T) \\ \tilde{Y}_{33} &= \tilde{P}_{33}/\sigma_a^2 \end{aligned} \tag{2.61}$$

Now evaluating (2.59), we get the following six nonlinear equations:

$$4H_1[r(2\tilde{Y}_{12} + \tilde{Y}_{13}) + 4(\tilde{Y}_{22} + \tilde{Y}_{23}) + \tilde{Y}_{33}] = H_2^2 \tag{2.62}$$

$$H_1(r\tilde{Y}_{13} + 4\tilde{Y}_{22} + 6\tilde{Y}_{23} + 2\tilde{Y}_{33}) = H_2(\tilde{Y}_{12} + \tilde{Y}_{13}) \tag{2.63}$$

$$2H_1(2\tilde{Y}_{23} + \tilde{Y}_{33}) = H_2\tilde{Y}_{13} \tag{2.64}$$

$$H_1(2\tilde{Y}_{23} + \tilde{Y}_{33}) = (\tilde{Y}_{12} + \tilde{Y}_{13})^2 \tag{2.65}$$

$$H_1\tilde{Y}_{33} = \tilde{Y}_{13}(\tilde{Y}_{12} + \tilde{Y}_{13}) \tag{2.66}$$

$$H_1 = \tilde{Y}_{13}^2 \tag{2.67}$$

where

$$\begin{aligned} H_1 &= (1 + \tilde{Y}_{11}) \\ H_2 &= [r\tilde{Y}_{11} + 2(2\tilde{Y}_{12} + \tilde{Y}_{13})] \end{aligned} \tag{2.68}$$

$$r = 4\sigma_x/(\sigma_a T^2) \tag{2.69}$$

The solution to the six nonlinear equations (2.62) to (2.67) is given separately in Appendix 2B. After considerable algebraic manipulations, the solution to nonlinear equations (2.62) to (2.67) may be obtained as

$$\begin{aligned}
\tilde{Y}_{11} &= m(m+2) \qquad (2.70)\\
\tilde{Y}_{12} &= rm^2/2\\
\tilde{Y}_{13} &= 1+m\\
\tilde{Y}_{22} &= r(3+2m)/4\\
\tilde{Y}_{23} &= rm/2\\
\tilde{Y}_{33} &= 2(1+m)/m
\end{aligned}$$

where the value of m is found by solving the cubic equation given by

$$rm^3 - 2(m^2 + 3m + 2) = 0 \qquad (2.71)$$

The solution of this cubic equation (2.71) is given separately in Appendix 2C.

The quantity r is a dimensionless parameter proportional to the ratio of the positional observation error σ_x to the position error caused by a constant acceleration of σ_a (ft/s^2), and hence may be regarded as the noise-to-signal ratio.

2.3.6 Steady State Gain Vector

Let the steady state gain vector K be defined as

$$K = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} \qquad (2.72)$$

Using (2.49), (2.60), and (2.67) in (2.56), the gain elements may be derived as:

$$\begin{aligned}
K_1 &= \tilde{Y}_{11}/\tilde{Y}_{13}^2 \qquad (2.73)\\
K_2 &= 4\tilde{Y}_{12}/rT\tilde{Y}_{13}^2\\
K_3 &= 4/rT^2\tilde{Y}_{13}
\end{aligned}$$

If the normalized gains are defined as

$$\begin{aligned}
G_1 &= K_1 \qquad (2.74)\\
G_2 &= TK_2\\
G_3 &= T^2K_3
\end{aligned}$$

From (2.73) and (2.70), (2.74) becomes

$$G_1 = m(m+2)/(1+m)^2 \tag{2.75}$$
$$G_2 = 2m^2/(1+m)^2$$
$$G_3 = 4/r(1+m)$$

2.3.7 Steady State $\hat{P}$ Matrix

Let the filtered covariance matrix be defined as

$$\hat{P} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & \hat{P}_{13} \\ \hat{P}_{12} & \hat{P}_{22} & \hat{P}_{23} \\ \hat{P}_{13} & \hat{P}_{23} & \hat{P}_{33} \end{bmatrix} \tag{2.76}$$

Then its normalized elements may be written as

$$\hat{Y}_{11} = \hat{P}_{11}/\sigma_x^2 \tag{2.77}$$
$$\hat{Y}_{12} = \hat{P}_{12}/(\sigma_x\sigma_a T)$$
$$\hat{Y}_{13} = \hat{P}_{13}/(\sigma_x\sigma_a)$$
$$\hat{Y}_{22} = \hat{P}_{22}/(\sigma_a^2 T^2)$$
$$\hat{Y}_{23} = \hat{P}_{23}/(\sigma_a^2 T)$$
$$\hat{Y}_{33} = \hat{P}_{33}/\sigma_a^2$$

Putting (2.49), (2.60), and (2.72) in (2.58), the $\hat{P}$ matrix may be found as

$$\hat{P} = \begin{bmatrix} (1-K_1)\tilde{P}_{11} & (1-K_1)\tilde{P}_{12} & (1-k_1)\tilde{P}_{13} \\ \tilde{P}_{12} - K_2\tilde{P}_{11} & \tilde{P}_{22} - K_2\tilde{P}_{12} & \tilde{P}_{23} - K_2\tilde{P}_{13} \\ \tilde{P}_{13} - K_3\tilde{P}_{11} & \tilde{P}_{23} - K_3\tilde{P}_{12} & \tilde{P}_{33} - K_3\tilde{P}_{13} \end{bmatrix} \tag{2.78}$$

Using (2.73), (2.77), and (2.67) in (2.78), the normalized elements of the $\hat{P}$ matrix may be derived as

$$\hat{Y}_{11} = \tilde{Y}_{11}/\tilde{Y}_{13}^2 \tag{2.79}$$
$$\hat{Y}_{12} = \tilde{Y}_{12}/\tilde{Y}_{13}^2$$
$$\hat{Y}_{13} = 1/\tilde{Y}_{13}$$
$$\hat{Y}_{22} = \tilde{Y}_{22} - \tilde{Y}_{12}^2/\tilde{Y}_{13}^2$$
$$\hat{Y}_{23} = \tilde{Y}_{23} - \tilde{Y}_{12}/\tilde{Y}_{13}$$
$$\hat{Y}_{33} = \tilde{Y}_{33} - 1$$

Using (2.70) in (2.79), we get

$$\begin{aligned}
\hat{Y}_{11} &= m(m+2)/(1+m)^2 \\
\hat{Y}_{12} &= rm^2/2(1+m)^2 \\
\hat{Y}_{13} &= 1/1+m \\
\hat{Y}_{22} &= r(3+m)/4(1+m) \\
\hat{Y}_{23} &= rm/2(1+m) \\
\hat{Y}_{33} &= (m+2)/m
\end{aligned} \tag{2.80}$$

2.3.8 Numerical Results

From (2.70), (2.75), and (2.80), it is seen that all the steady state normalized covariances and gains are functions of the dimensionless parameter r. As an example for

$$r = 0.90 \text{ (say)}$$

the steady state $\tilde{Y}$, $\hat{Y}$, and G matrices may be evaluated from Eqs. (2.70), (2.80), and (2.75) and the results are given below.

$$\tilde{Y} = \begin{bmatrix} 25.0946 & 7.5951 & 5.1083 \\ 7.5951 & 2.5237 & 1.8487 \\ 5.1083 & 1.8487 & 2.4868 \end{bmatrix}$$

$$\hat{Y} = \begin{bmatrix} 0.9617 & 0.2911 & 0.1958 \\ 0.2911 & 0.3131 & 0.3619 \\ 0.1958 & 0.3619 & 1.4868 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.9617 \\ 1.2936 \\ 0.8700 \end{bmatrix}$$

The normalized covariances before and after position determination are plotted against r in Figures 2.4 to 2.6, and the normalized velocity and acceleration gains are plotted in Figures 2.7 and 2.8 against r.

For evaluating the unnormalized covariances and gains, let

$$T = 4 \text{ s}$$

$$\sigma_a = 0.01414 \text{ nm/s}^2$$

then from (2.69), for $r = 0.9$, we get

$$\sigma_x = 0.0509$$

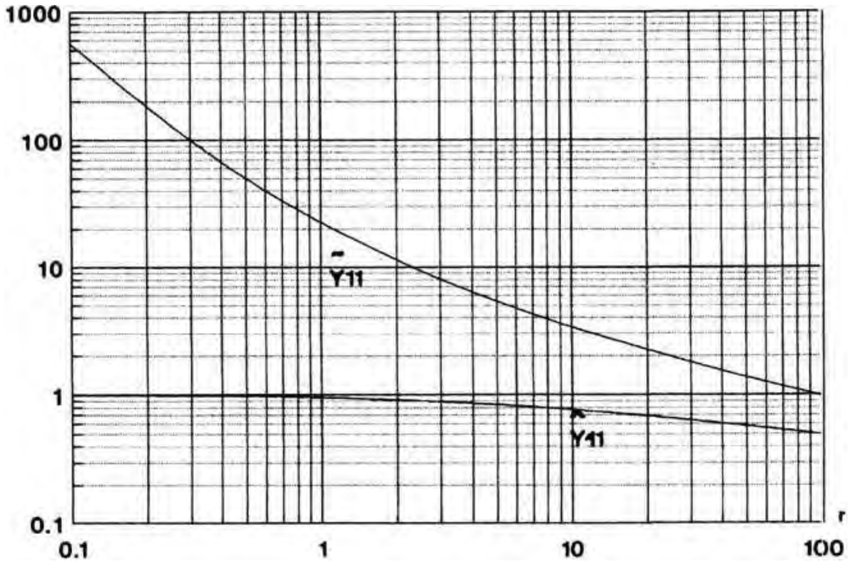

Figure 2.4 Position accuracy before and after measurements.

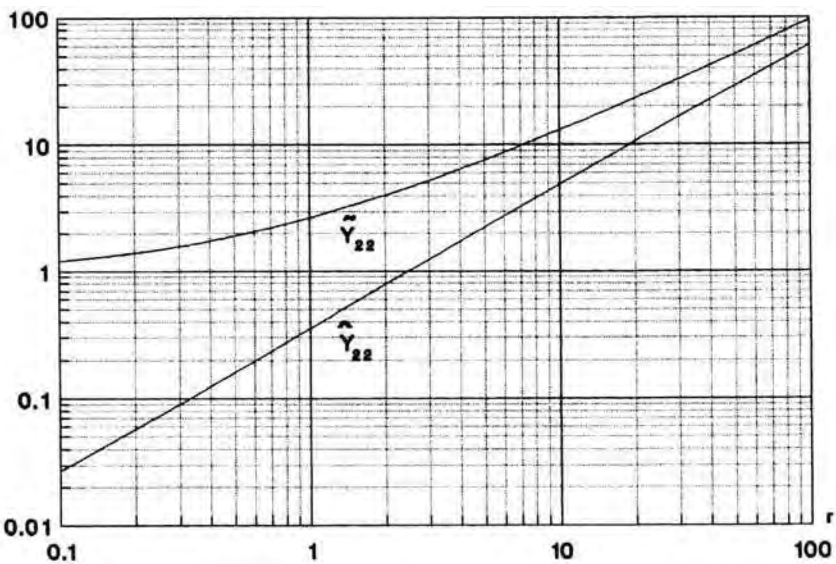

Figure 2.5 Velocity accuracy before and after position determination.

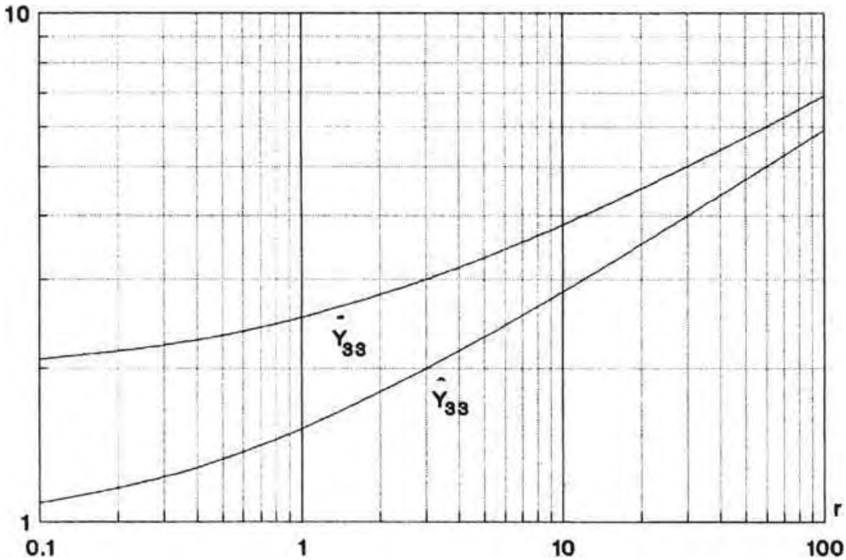

Figure 2.6 Acceleration accuracy before and after position determination.

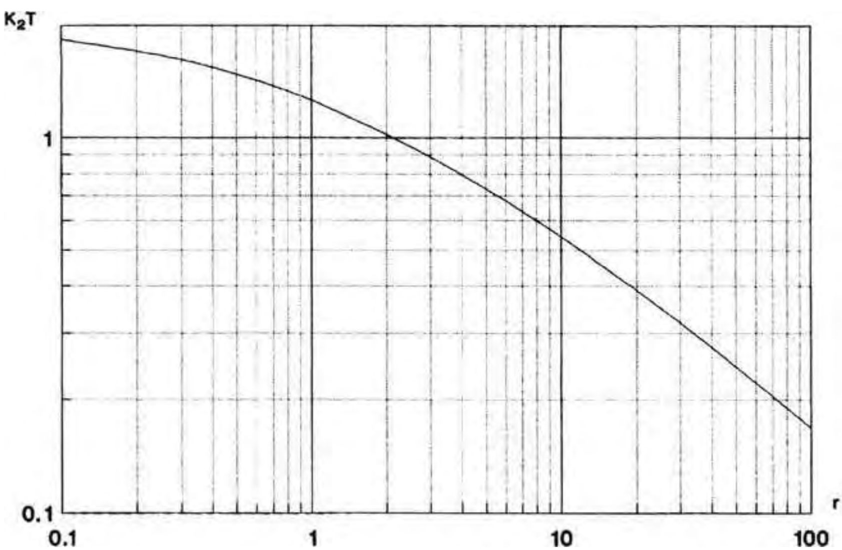

Figure 2.7 Normalized velocity gain K_2T as a function of r.

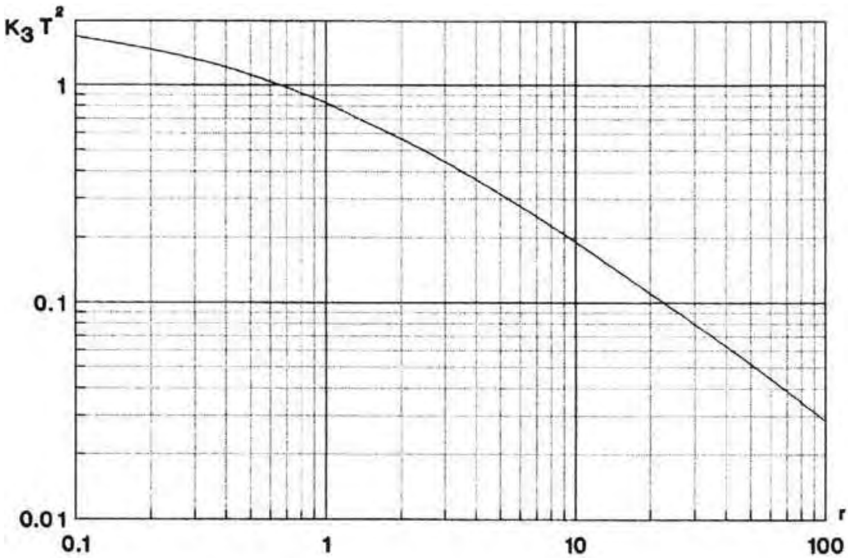

Figure 2.8 Normalized acceleration gain as a function of r.

As $\tilde{Y}$, $\hat{Y}$, and G are known for these parameters, $\tilde{P}$, $\hat{P}$, and K matrices may be evaluated from (2.61), (2.77), and (2.74) as

$$\tilde{P} = \begin{bmatrix} 0.0650 & 0.0219 & 0.0037 \\ 0.0219 & 0.0081 & 0.0015 \\ 0.0037 & 0.0015 & 0.0005 \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} 0.0025 & 0.0008 & 0.0001 \\ 0.0008 & 0.0010 & 0.0003 \\ 0.0001 & 0.0003 & 0.0003 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.9617 \\ 5.1744 \\ 13.9207 \end{bmatrix}$$

If $\tilde{P}$, $\hat{P}$, and K matrices are evaluated by executing the Kalman filter matrix equations (2.56) to (2.58) recursively, we get the same values as given above.

2.3.9 Mean Square Values of the Ripple

The mean square values of the ripple in position, velocity, and acceleration are given by

$$\frac{\tilde{P}_{11} - \hat{P}_{11}}{\sigma_x^2} = \frac{m^2(m+2)^2}{(1+m)^2}$$

$$\frac{\tilde{P}_{22} - \hat{P}_{22}}{\sigma_a^2 T^2} = \frac{rm(m+2)}{2(1+m)}$$

$$\frac{\tilde{P}_{33} - \hat{P}_{33}}{\sigma_a^2} = 1$$

2.3.10 Design Formula for Sampling Time

From Figure 2.4, it is seen that $\hat{Y}_{11} \leq 1$, whereas $\tilde{Y}_{11}$ can be larger or smaller than unity depending upon the level of plant noise. From this figure, it is seen that the crossover point occurs for $r \simeq 96$. From this we obtain the relation

$$T < 0.2041\sqrt{\sigma_x/\sigma_a} \tag{2.81}$$

as the suitable sampling time which would keep the position error in a sampled data system below the inherent sensor error [2].

2.4 A THREE-STATE FILTER: RAMACHANDRA'S MODEL II

Ramachandra's model II [3] is also a one-dimensional dynamic model for estimating the optimum steady state position, velocity, and acceleration of an aircraft or similar vehicle moving with a constant acceleration and acted upon by a zero mean plant noise which perturbs its constant acceleration motion and accounts for maneuvers and/or other random factors. As in the previous two models, each position coordinate of the vehicle is assumed to be measured by a track-while-scan radar at uniform intervals of time T seconds through random noise.

2.4.1 Dynamic Model

For each position coordinate of the vehicle, the dynamics of the target may be represented as

$$\begin{aligned} x_{n+1} &= x_n + \dot{x}_n T + \ddot{x}_n T^2/2 + a_n T^3/6 \\ \dot{x}_{n+1} &= \dot{x}_n + \ddot{x}_n T + a_n T^2/2 \\ \ddot{x}_{n+1} &= \ddot{x}_n + a_n T \end{aligned} \tag{2.82}$$

where

x_n = vehicle position at scan n
$\dot{x}_n$ = vehicle velocity at scan n
$\ddot{x}_n$ = vehicle acceleration at scan n
a_n = plant noise (rate of change of acceleration in ft/s^3) at scan n
T = sampling time

It is assumed that the plant noise is a random constant between successive observations having zero mean and constant variance σ_a^2 and also uncorrelated with its values at other inrevals; i.e., a_n satisfies the statistical properties (2.2).

In vector-matrix form, Eqs. (2.82) can be written as

$$X_{n+1} = FX_n + Ga_n \tag{2.83}$$

where X_n and F are as defined in (2.45) and (2.46). G is given by

$$G = \begin{bmatrix} T^3/6 \\ T^2/2 \\ T \end{bmatrix} \tag{2.84}$$

2.4.2 Measurement Equation

The measurement equation is given by (2.48).

2.4.3 Filtering Equations

The filtering equations are given by (2.11) to (2.15). $Q = \sigma_a^2$ is the variance of the plant noise.

2.4.4 Steady State $\tilde{P}$ Matrix

If the covariance matrix $\tilde{P}$ is defined as 3×3 symmetric matrix given by (2.60), the normalized covariances for this model may be written as

$$\tilde{Y}_{11} = \tilde{P}_{11}/\sigma_x^2 \tag{2.85}$$

$$\tilde{Y}_{12} = 2\tilde{P}_{12}/(\sigma_x\sigma_a T^2)$$

$$\tilde{Y}_{13} = \tilde{P}_{13}/(\sigma_x\sigma_a T)$$

$$\tilde{Y}_{22} = 12\tilde{P}_{22}/(\sigma_a^2 T^4)$$

$$\tilde{Y}_{23} = 2\tilde{P}_{23}/(\sigma_a^2 T^3)$$

$$\tilde{Y}_{33} = \tilde{P}_{33}/(\sigma_a^2 T^2)$$

Evaluating the combined covariance equation (2.20) for this model, we get the following six nonlinear equations:

$$4H_1[1 + 3r(\tilde{Y}_{12} + \tilde{Y}_{13}) + 3\tilde{Y}_{22} + 18\tilde{Y}_{23} + 9\tilde{Y}_{33}] = H_2^2 \tag{2.86}$$

$$2H_1(1 + r\tilde{Y}_{13} + \tilde{Y}_{22} + 9\tilde{Y}_{23} + 6\tilde{Y}_{33}) = H_2(\tilde{Y}_{12} + 2\tilde{Y}_{13}) \tag{2.87}$$

$$2H_1[1 + 3(\tilde{Y}_{23} + \tilde{Y}_{33})] = \tilde{Y}_{13}H_2 \tag{2.88}$$

$$H_1[1 + 4(\tilde{Y}_{23} + \tilde{Y}_{33})] = (\tilde{Y}_{12} + 2\tilde{Y}_{13})^2 \tag{2.89}$$

$$H_1[1 + 2\tilde{Y}_{33}) = \tilde{Y}_{13}(\tilde{Y}_{12} + 2\tilde{Y}_{13}) \tag{2.90}$$

$$H_1 = \tilde{Y}_{13}^2 \tag{2.91}$$

where

$$H_1 = 1 + \tilde{Y}_{11} \tag{2.92}$$

$$H_2 = r\tilde{Y}_{11} + 6(\tilde{Y}_{12} + \tilde{Y}_{13}) \tag{2.93}$$

$$r = 12\sigma_x/(\sigma_a T^3) \tag{2.94}$$

The quantity r is a dimensionless parameter proportional to the ratio of the positional observation error σ_x to the position error caused by a constant change of acceleration of σ_a (ft/s^3) and hence may be regarded as the noise-to-signal ratio [3].

The solution to the six nonlinear equations (2.86) to (2.91) is given separately in Appendix 2D.

After considerable algebraic manipulations, the steady state predicted covariances may be found as

$$\tilde{Y}_{11} = 2S(S_1 + S)/r^2 \tag{2.95}$$
$$\tilde{Y}_{12} = S_2(S_1 + S)/Ar$$
$$\tilde{Y}_{13} = (S_1 + S)/r$$
$$\tilde{Y}_{22} = S_2(A^2 + 2S + AS_2)/(2A) - S_1$$
$$\tilde{Y}_{23} = (S_2 + A)^2/12$$
$$\tilde{Y}_{33} = (S_2 + A)/2A$$

where

$$S_1 = \sqrt{S^2 + r^2} \tag{2.96}$$
$$S_2 = \sqrt{4S - 1}$$
$$A = \sqrt{3}$$

S is obtained by solving the biquadratic equation given by

$$S^4 - 6S^3 + 10S^2 - 6mS + n = 0 \tag{2.97}$$

where m and n are given by

$$m = 1 + 2r^2$$
$$n = 1 + 3r^2$$

The solution to this biquadratic equation may be found as:

$$S = \tfrac{1}{2}(a + \sqrt{a^2 - 4b}) \tag{2.98}$$
$$a = 3 + \sqrt{2z - 1} \tag{2.99}$$
$$b = z - \sqrt{z^2 - (1 + 3r^2)}$$
$$z = U - V + \tfrac{5}{3}$$
$$U = (D - C)^{1/3}$$
$$V = (D + C)^{1/3}$$
$$D = r\sqrt{\left[\tfrac{1}{2} + r^2\left\{\tfrac{145}{16} + r^2\left(\tfrac{97}{2} + 81r^2\right)\right\}\right]}$$
$$C = r^2\left[\tfrac{17}{4} - 9r^2\right] + \tfrac{1}{27}$$

2.4.5 Steady State Gain Vector

If the steady state gain vector K is defined as in (2.72), then its normalized elements may written as given in (2.74). The normalized elements of K for this model may be derived as

$$G_1 = 2S(S_1 - S)/r^2 \tag{2.100}$$

$$G_2 = 2AS_2(S_1 - S)/r^2$$

$$G_3 = 12(S_1 - S)/r^2$$

2.4.6 Steady State Covariance Matrix $\hat{P}$

The normalized elements of $\hat{P}$ matrix may be found as

$$\hat{Y}_{11} = 2S(S_1 - S)/r^2 \tag{2.101}$$

$$\hat{Y}_{12} = S_2(S_1 - S)/Ar$$

$$\hat{Y}_{13} = (S_1 - S)/r$$

$$\hat{Y}_{22} = S_2(A^2 + 2S - AS_2)/(2A) - S_1$$

$$\hat{Y}_{23} = (S_2 - A)^2/12$$

$$\hat{Y}_{33} = (S_2 - A)/(2A)$$

where $\hat{Y}_{ij}$ are as defined in (2.85) by replacing tildes by hats on both sides.

2.4.7 Numerical Results

From (2.95), (2.101), and (2.100), it is seen that all the steady state normalized covariances and gains are functions of the dimensionless parameter r. As an example, for

$$r = 4.00 \text{ (say)}$$

the steady state $\tilde{Y}$, $\hat{Y}$, and G matrices may be evaluated from Eqs. (2.95), (2.101), and (2.100), and the results may be found as follows:

$$\tilde{Y} = \begin{bmatrix} 16.3186 & 13.2410 & 4.1616 \\ 13.2410 & 36.1068 & 4.3717 \\ 4.1616 & 4.3717 & 2.0909 \end{bmatrix}$$

$$\hat{Y} = \begin{bmatrix} 0.9423 & 0.7646 & 0.2403 \\ 0.7646 & 5.7366 & 1.1900 \\ 0.2403 & 1.1900 & 1.0909 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.9423 \\ 1.1468 \\ 0.7209 \end{bmatrix}$$

The normalized covariances before and after position determination and the normalized velocity and acceleration gains are plotted in Figures 2.9 to 2.13 against r.

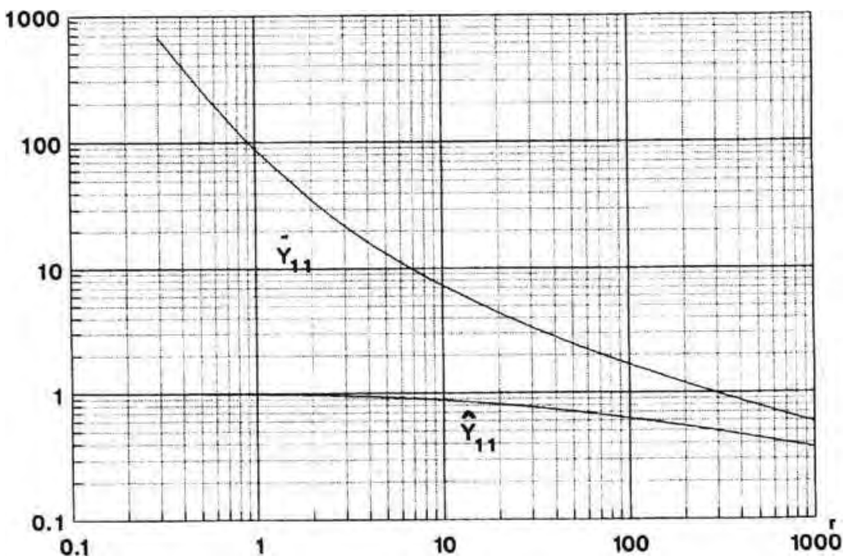

Figure 2.9 Position accuracy before and after position determination. (From Ref. 3; © 1987—IEEE.)

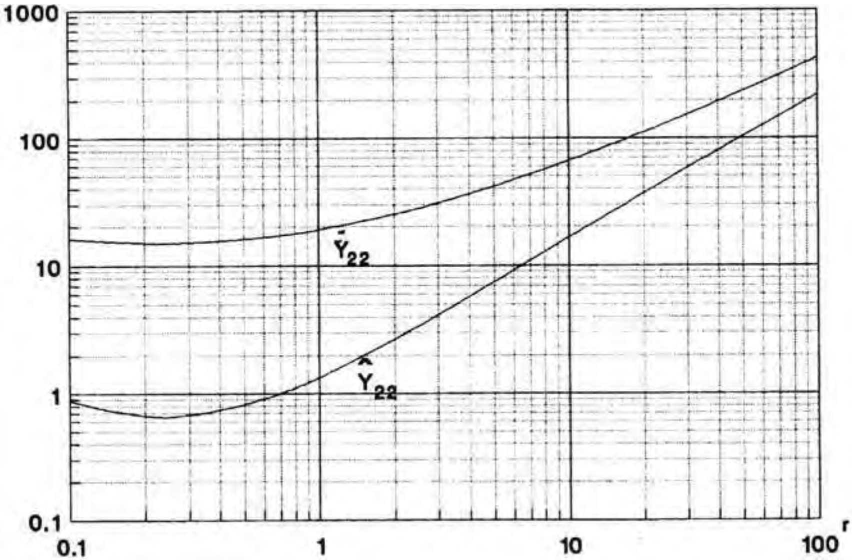

Figure 2.10 Velocity accuracy before and after position determination.

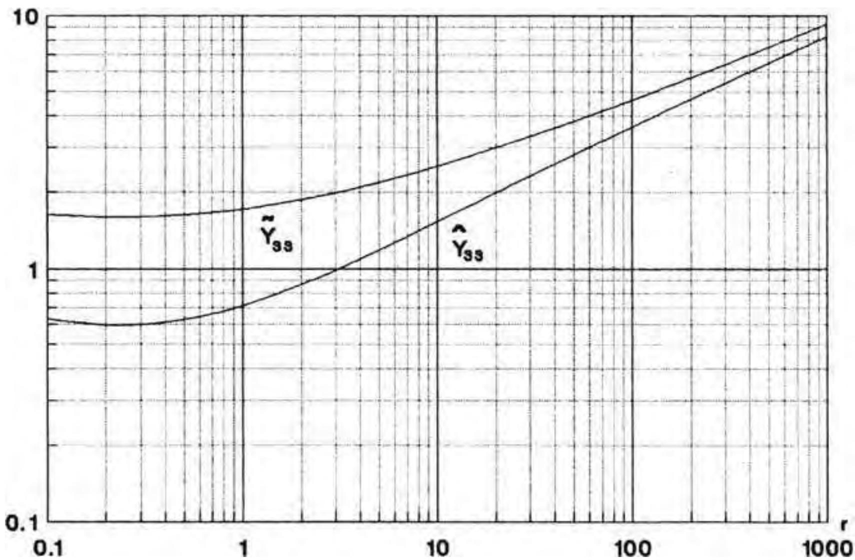

Figure 2.11 Acceleration accuracy before and after position determination.

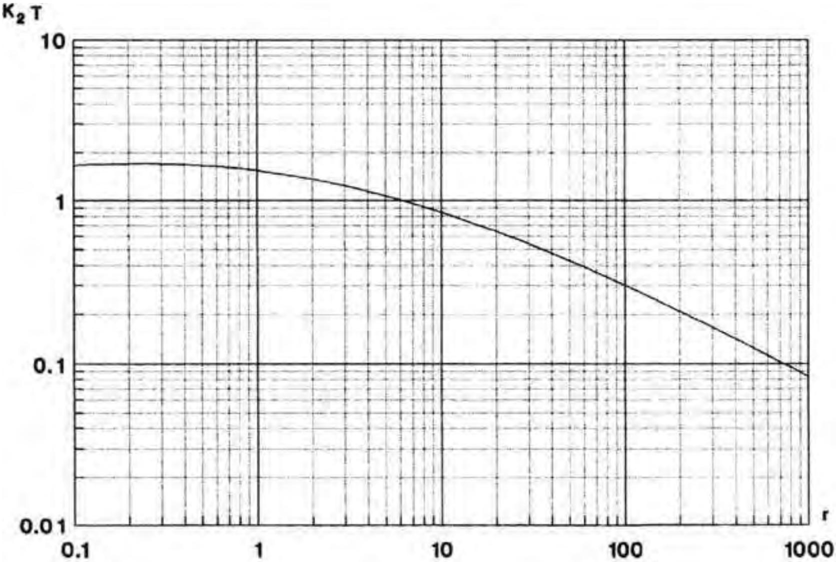

Figure 2.12 Normalized velocity gain K_2T as a function of r.

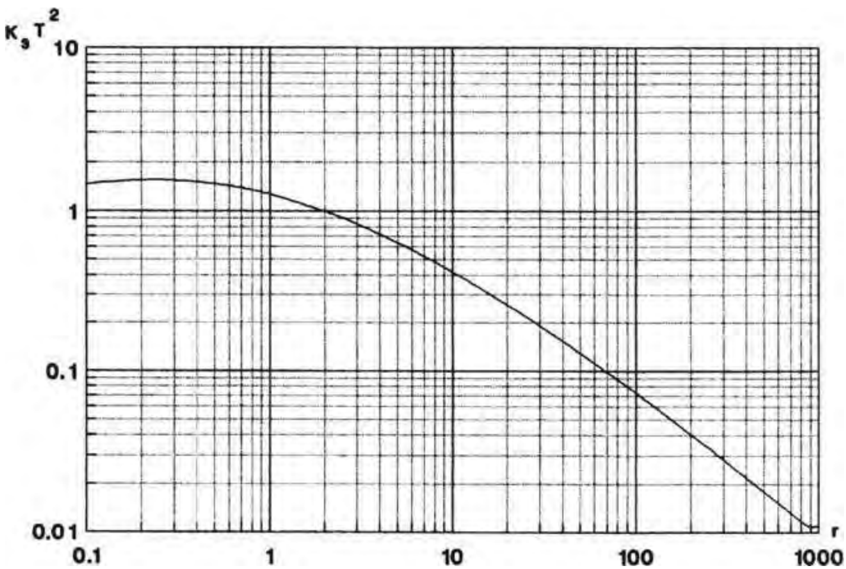

Figure 2.13 Normalized acceleration gain K_3T^2 as a function of r.

For evaluating the unnormalized covariances and gains, let

$$T = 4 \text{ s}$$
$$\sigma_a = 0.01414 \text{ nm/s}^3$$

then from (2.94), for $r = 4.0$, we get

$$\sigma_x = 0.3017 \text{ nm}$$

Knowing $\tilde{Y}$, $\hat{Y}$, and G matrices for these parameters, $\tilde{P}$, $\hat{P}$, and K matrices may be evaluated as

$$\tilde{P} = \begin{bmatrix} 1.4849 & 0.9036 & 0.0710 \\ 0.9036 & 1.8481 & 0.0559 \\ 0.0710 & 0.0559 & 0.0067 \end{bmatrix}$$
$$\hat{P} = \begin{bmatrix} 0.0857 & 0.0522 & 0.0041 \\ 0.0522 & 0.2936 & 0.0152 \\ 0.0041 & 0.0152 & 0.0035 \end{bmatrix}$$
$$K = \begin{bmatrix} 0.9423 \\ 4.5873 \\ 11.5341 \end{bmatrix}$$

If $\tilde{P}$, $\hat{P}$, and K matrices are evaluated by executing the Kalman filter matrix equations (2.17) to (2.19) recursively to the steady state, we get the same values as given above.

2.4.8 Mean Square Values of the Ripple

The mean square values of the ripple in position, velocity, and acceleration are given by

$$\frac{\tilde{P}_{11} - \hat{P}_{11}}{\sigma_x^2} = \frac{4S^2}{r^2} \tag{2.102}$$
$$\frac{\tilde{P}_{22} - \hat{P}_{22}}{\sigma_a^2 T^4} = \frac{S_2^2}{12}$$
$$\frac{\tilde{P}_{33} - \hat{P}_{33}}{\sigma_a^2 T^2} = 1$$

2.4.9 Design Formula for Sampling Time

From Figure 2.9, it is seen that $\hat{Y}_{11} \leq 1$, whereas $\tilde{Y}_{11}$ can be larger or smaller than unity depending upon the level of plant noise. From this figure, it is seen that the crossover point occurs for $r \simeq 287.62$. From this we obtain the relation

$$T < 0.3468[\sigma_x/\sigma_a]^{1/3} \tag{2.103}$$

as the suitable sampling time which would keep the position error in a sampled data system below the inherent sensor error [3].

2.5 SUMMARY

A one-dimensional two-state model of Friedland [1] for estimating the position and velocity of an aircraft or similar vehicle moving in straight line path perturbed by a zero mean random acceleration is discussed in Section 2.2. Two models for estimating position, velocity, and acceleration based on Kalman filtering techniques are discussed in Sections 2.3 and 2.4. In all three models, each position coordinate of the vehicle is assumed to be measured independently in the cartesian coordinate system. The steady state characteristics of the models are analytically obtained by directly solving the discrete Riccati equation.

Bridgewater [4] has presented an analysis of second- and third-order steady state tracking filters. A general algorithm is presented for recursively computing Kalman gains. Changes in sampling time and variances of measurement noise and target maneuver can be readily incorporated in the computation of gains. Steady state expressions for $\alpha - \beta$ and $\alpha - \beta - \gamma$ filters are presented in Ref. 4.

The two-state filter requires two measurements for its initialization, and three measurements are required for initializing the three-state filter.

The steady state results presented here for a single physical dimension can be readily applied for the range-bearing, range-bearing-elevation, and bearing-elevation sensor measurement sets provided by any of radar, sonar, and IR sensors.

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APPENDIX 2A: SOLUTION OF NONLINEAR EQUATIONS OF FRIEDLAND'S MODEL

Putting (2.24) and (2.25) in (2.23),

$$\frac{4}{r}\tilde{Y}_{22}^2(2\tilde{Y}_{22}+1) = \frac{4}{r}\tilde{Y}_{22}^4 - r + \frac{8}{r}\tilde{Y}_{22}^2$$

Multiplying throughout by r,

$$4\tilde{Y}_{22}^2(2\tilde{Y}_{22}+1) = 4\tilde{Y}_{22}^4 - r^2 + 8\tilde{Y}_{22}^2 \tag{A1}$$

$$4\tilde{Y}_{22}^4 - 4\tilde{Y}_{22}^2(2\tilde{Y}_{22}+1) + 8\tilde{Y}_{22}^2 = r^2$$

$$4\tilde{Y}_{22}^2(\tilde{Y}_{22}^2 - 2\tilde{Y}_{22} + 1) = r^2$$

$$4\tilde{Y}_{22}^2(\tilde{Y}_{22} - 1)^2 = r^2$$

$$2\tilde{Y}_{22}^2 - 2\tilde{Y}_{22} - r = 0$$

$$\tilde{Y}_{22} = (1 + \sqrt{1+2r})/2$$

From (2.24) and (2.25),

$$\tilde{Y}_{12} = (1 + \sqrt{1+2r})^2/(2r) \tag{A2}$$

$\tilde{Y}_{11}$ can be obtained from (2.25) and (A2).

APPENDIX 2B: SOLUTION OF SIX NONLINEAR EQUATIONS OF RAMACHANDRA'S MODEL I

From (2.67),

$$\tilde{Y}_{11} = \tilde{Y}_{13}^2 - 1 \tag{B1}$$

Using (2.67) in (2.66),

$$\tilde{Y}_{12} + \tilde{Y}_{13} = \tilde{Y}_{13}\tilde{Y}_{33} \tag{B2}$$

or

$$\tilde{Y}_{33} = 1 + Y_{12}/Y_{13} \tag{B3}$$

Using (2.67) in (2.65),

$$2\tilde{Y}_{23} = \tilde{Y}_{33}(\tilde{Y}_{33} - 1) \tag{B4}$$

Dividing (2.63) by (2.64) and simplifying using (B2) and (B4),

$$r\tilde{Y}_{13} + 4\tilde{Y}_{22} = 2\tilde{Y}_{23}(2\tilde{Y}_{33} - 1) \tag{B5}$$

Squaring (2.64) and dividing by (2.62) and simplifying using (B2),(B4), and (B5),

$$r\tilde{Y}_{12} = 2\tilde{Y}_{23}^2 \tag{B6}$$

From (2.64), (2.67), and (B4),

$$H_2 = 2\tilde{Y}_{13}\tilde{Y}_{33}^2 \tag{B7}$$

Dividing (2.64) by (2.65) and simplifying using (B1), (B7), and (2.68), we get

$$\tilde{Y}_{12}^2 = \frac{r}{2}\tilde{Y}_{13}(\tilde{Y}_{13}^2 - 1) \tag{B8}$$

From (2.68), (B1), (B2), (B6), and (B7), we get the following quadratic in $\tilde{Y}_{23}$:

$$4\tilde{Y}_{23}^2 - 4r\tilde{Y}_{13}\tilde{Y}_{23} + r^2(\tilde{Y}_{13}^2 - 1) = 0 \tag{B9}$$

Solving (B9), the two roots of $\tilde{Y}_{23}$ may be found as:

$$\tilde{Y}_{23} = \frac{r}{2}(\tilde{Y}_{13} - 1) \tag{B10}$$

$$\tilde{Y}_{23} = \frac{r}{2}(\tilde{Y}_{13} + 1) \tag{B11}$$

Putting (B10) in (B6),

$$\tilde{Y}_{12} = \frac{r}{2}(\tilde{Y}_{13} - 1)^2 \tag{B12}$$

Let

$$m = \tilde{Y}_{13} - 1 \tag{B13}$$

Putting (B13) in (B12),

$$\tilde{Y}_{12} = \frac{rm^2}{2} \tag{B14}$$

Putting (B13) in (B8),

$$\tilde{Y}_{12}^2 = \frac{r}{2} m(m+1)(m+2) \tag{B15}$$

Putting the value of $\tilde{Y}_{12}$ from (B14) in (B15), we get

$$rm^3 = 2(m+1)(m+2) \tag{B16}$$

(B16) is the cubic equation given by (2.71). m is obtained by solving (B16). The solution of this cubic is given in Appendix 2C.

Putting (B13) in (B1), $\tilde{Y}_{11}$ is obtained. $\tilde{Y}_{12}$ is obtained from (B14), and $\tilde{Y}_{13}$ from (B13). Putting (B13) in (B10), $\tilde{Y}_{23}$ is obtained.

Putting the values of $\tilde{Y}_{12}$ and $\tilde{Y}_{13}$ from (B14) and (B13) in (B3), we get

$$\tilde{Y}_{33} = \frac{rm^2}{2(1+m)} \tag{B17}$$

From (B16),

$$rm^2 = 2(m+1)(m+2) \tag{B18}$$

Putting (B18) in (B17), $\tilde{Y}_{33}$ may also be expressed as given in (2.70). $\tilde{Y}_{22}$ is obtained from (B5).

Thus all predicted covariances are determined as given in (2.70).

The other root of $\tilde{Y}_{23}$ given by (B11) is discarded since its value does not tally with that obtained by executing the Kalman filter matrix equations.

APPENDIX 2C: SOLUTION OF THE CUBIC EQUATION

The solution of the cubic equation (2.71) may be obtained as follows: Let

$$E = 27r^2$$

If $E > 1$, then m is given by

$$m = a + b + \frac{2}{3r}$$

where

$$a = (c + d)^{1/3}$$

$$b = (c - d)^{1/3}$$

with

$$c = \frac{2}{r}\left(1 + \frac{1}{r} + \frac{4}{E}\right)$$

$$d = \frac{2}{r}\sqrt{1 - \frac{1}{E}}$$

If $E < 1$, then m is given by

$$m = 2\left[\frac{1}{3r} + s\cos\left(\frac{h}{3}\right)\right]$$

where

$$s = \sqrt{\frac{2}{r}\left(1 + \frac{2}{9r}\right)}$$

$$h = \tan^{-1}\left(\frac{d'}{c}\right)$$

$$d' = \frac{2}{r}\sqrt{\frac{1}{E} - 1}$$

If $E = 1$, then m is given by

$$m = 2\left(c^{1/3} + \frac{1}{3r}\right)$$

APPENDIX 2D: SOLUTION OF SIX NONLINEAR EQUATIONS OF RAMACHANDRA'S MODEL II

The solution to (2.86) to (2.91) is obtained as follows. From (2.90) and (2.91), we get

$$\tilde{Y}_{12} + \tilde{Y}_{13} = 2\tilde{Y}_{13}\tilde{Y}_{33} \tag{D1}$$

From (2.89) to (2.91), we get

$$\tilde{Y}_{23} = \tilde{Y}_{33}^2 \tag{D2}$$

From (2.88), (D1), (D2), and (2.91), we get after simplification

$$\tilde{Y}_{11} = \frac{2S(S_1 + S)}{r^2} \tag{D3}$$

$$\tilde{Y}_{13} = \frac{S_1 + S}{r} \tag{D4}$$

where

$$S = 3\tilde{Y}_{33}^2 - 3\tilde{Y}_{33} + 1 \tag{D5}$$

From (2.88) and (2.90), we get

$$\tilde{Y}_{12} = \frac{S_2\tilde{Y}_{13}}{A} \tag{D6}$$

From (D1) and (D6),

$$\tilde{Y}_{33} = \frac{S_2 + A}{2A} \tag{D7}$$

From (2.87),

$$\tilde{Y}_{22} = 6\tilde{Y}_{33}^3 - \tilde{Y}_{33} - r\tilde{Y}_{13} \tag{D8}$$

Using the above results, we get from (2.86),

$$S^4 - 6S^3 + 10S^2 - 6mS + n = 0 \tag{D9}$$

where

$$m = 1 + 2r^2 \tag{D10}$$
$$n = 1 + 3r^2$$

Solving the biquadratic (D9), the value of S is determined. $\tilde{Y}_{33}$ is found from (D5). $\tilde{Y}_{11}$ and $\tilde{Y}_{13}$ are determined from (D3) and (D4). $\tilde{Y}_{12}$ is then found from (D6). $\tilde{Y}_{22}$ and $\tilde{Y}_{23}$ are found using (D8) and (D2).

3

Discrete-Time Two-Dimensional Tracking Filters

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3.1 INTRODUCTION

The uncoupled one-dimensional tracking filters described in Chapter 2 may be extended to two dimensions to develop two-dimensional trackers for estimating the position, velocity, and also acceleration of an aircraft with the range r and bearing θ measurements obtained by a two-dimensional track-while-scan radar sensor. The techniques and matrix transformations to develop two-dimensional trackers are presented in this chapter.

The measurements obtained at discrete intervals of time T seconds are assumed to be corrupted with range noise and angular noise. The tracking operation is assumed to be performed in the cartesian coordinate system. The coupling between the quantities measured by the radar (r, θ) and the two-dimensional cartesian coordinate system selected for tracking operation is explicitly considered in the development of the two-dimensional models.

The steady state filter characteristics of the two-dimensional trackers are analytically determined making use of the properties of the uncoupled one-dimensional trackers discussed in Chapter 2. These results are of practical interest in developing trackers for tracking aircraft and similar vehicles. These results also eliminate the real time execution of the complete Kalman filter matrix equations, providing a significant reduction in tracking and updating time. This is illustrated in the extension of the one-dimensional Friedland's model and Ramachandra's model I to two dimensions.

3.2 TECHNIQUES AND MATRIX TRANSFORMATIONS

In Ref. 1, Fitzgerald gives the techniques and also the necessary matrix transformation equations for developing two-dimensional tracking filters. Using these techniques, the steady state results of the two-dimensional filters may be expressed in a concise form. The technique involves the determination of covariances and gains in an uncoupled system and then transforms them for application in the coupled system.

3.2.1 Two-Dimensional Two-State Filters

Consider the case of tracking an aircraft or similar vehicle in the two-dimensional cartesian coordinate system with two-state filters. Let the state vector be represented as

$$X^T = [x \quad y \quad \dot{x} \quad \dot{y}] \tag{3.1}$$

A two-dimensional track-while-scan radar sensor is assumed to measure the range r and bearing θ of the vehicle at uniform sampling intervals of time T seconds, and all measurements are assumed to be corrupted with range noise and angular noise.

If P_0 and K_0 denote the covariance matrix and the corresponding filter gain matrix for tracking along the x axis corresponding to $\theta = 0$, then for tracking at an arbitrary angle θ, Fitzgerald [1] shows that the covariances and gain matrices may be expressed as

$$P = A_{22} P_0 A_{22}^T \tag{3.2}$$

$$K = A_{22} K_0 A_2^T \tag{3.3}$$

where

$$A_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{3.4}$$

$$A_{22} = \begin{bmatrix} A_2 & 0 \\ 0 & A_2 \end{bmatrix} \tag{3.5}$$

A_2 is the rotational matrix and A_{22} is a 4×4 matrix used for two dimensions and two-state filters. Equation (3.2) is valid for both predicted and filtered covariance matrices $\tilde{P}$ and $\hat{P}$.

The application of this technique for two-dimensional two-state filters is presented in Section 3.3 in the extension of Friedland's model to two dimensions.

3.2.2 Two-Dimensional Three-State Filters

For three-state filters in two dimensions, the state vector is assumed to be arranged as

$$X^T = [x \quad y \quad \dot{x} \quad \dot{y} \quad \ddot{x} \quad \ddot{y}] \tag{3.6}$$

If P_0 and K_0 are the covariance matrix and the corresponding Kalman filter gain matrix for tracking along the x axis corresponding to $\theta = 0$, then for tracking at an arbitrary angle θ, Fitzgerald [1] shows that the covariances and gain matrices may be expressed as:

$$P = A_{23} P_0 A_{23}^T \tag{3.7}$$

$$K = A_{23} K_0 A_2^T \tag{3.8}$$

where

$$A_{23} = \begin{bmatrix} A_2 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_2 \end{bmatrix} \tag{3.9}$$

A_2 is the rotational matrix given in (3.4) and A_{23} is a 6×6 matrix used for two-dimensional three-state filters. Equation (3.7) holds good for both predicted and filtered covariance matrices $\tilde{P}$ and $\hat{P}$.

When (3.2) and (3.3) or (3.7) and (3.8) are used for tracking in the two-dimensional cartesian coordinate system for any bearing angle θ, the tracking will still be nearly optimum provided the rate of change of bearing is slow [1].

This technique for two-dimensional three-state filters is applied for extension of Ramachandra's model I to two dimensions in Section 3.4.

3.3 CASTELLA-DUNNEBACKE'S MODEL: AN EXTENSION OF FRIEDLAND'S MODEL TO TWO DIMENSIONS

In this section, a two-dimensional tracking filter developed by Castella and Dunnebacke [2] for estimating the position and velocity of an aircraft or similar vehicle is discussed. The vehicle is assumed to be moving with a constant velocity motion perturbed by a zero mean random acceleration. The vehicle range r and bearing θ are assumed to be measured by a two-dimensional track-while-scan radar sensor at uniform sampling intervals of time T seconds and all measurements are noisy.

In this model, the coupling between the quantities measured by the radar and the cartesian xy coordinate system selected for tracking operation is explicitly considered. The steady state characteristics of the filter are analytically determined under the assumption of a white noise maneuver model in two dimensions. This model is an extension of Friedland's model to two dimensions.

3.3.1 Dynamic Model

In two dimensions, the vehicle dynamics may be represented by the vector-matrix equation of the form

$$X_{n+1} = FX_n + Ga_n \tag{3.10}$$

where

$$X_n^T = [x_n \quad y_n \quad \dot{x}_n \quad \dot{y}_n] \tag{3.11}$$

$$F = \begin{bmatrix} I & (T)I \\ 0 & I \end{bmatrix} \tag{3.12}$$

$$G = \begin{bmatrix} (T^2/2)I \\ (T)I \end{bmatrix} \tag{3.13}$$

where I is a 2×2 identity matrix and 0 is a 2×2 null matrix. F is a 4×4 matrix and G is a 4×2 matrix. a_n is the random acceleration acting on the vehicle with variance $Q = \sigma_a^2$. The random acceleration is assumed to be of equal variance and also independent along the x and y axes. This maneuver model also assumes that the acceleration along the x or y axis is a random constant between successive scans with zero mean and constant variance σ_a^2. Acceleration values at different scans are assumed to be uncorrelated (white noise maneuver model).

3.3.2 Measurement Model

The measurement equation may be written as

$$Z_n = HX_n + V_n \tag{3.14}$$

where

$$Z_n = \begin{bmatrix} x_m(n) \\ y_m(n) \end{bmatrix} \tag{3.15}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{3.16}$$

and

$$V_n = \begin{bmatrix} v_x(n) \\ v_y(n) \end{bmatrix} \tag{3.17}$$

$x_m(n)$ = measured x coordinate at scan n
$y_m(n)$ = measured y coordinate at scan n
$v_x(n)$ = random noise on x measurement at scan n
$v_y(n)$ = random noise on y measurement at scan n

From the tracking geometry illustrated in Figure 3.1,

$$\begin{aligned} x_n &= r(n)\cos\theta(n) \\ y_n &= r(n)\sin\theta(n) \end{aligned} \tag{3.18}$$

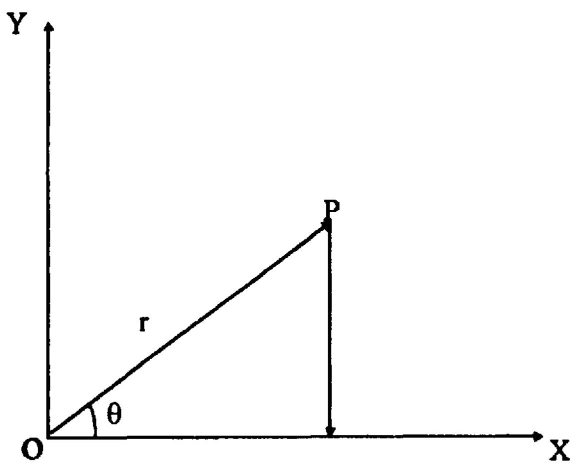

Figure 3.1 Two-dimensional tracking geometry.

As the measurements are in polar coordinates and tracking is done in cartesian coordinates, the measurements are coupled. The covariance matrix of the measurement noise V_n may be written as

$$R_n = \begin{bmatrix} \sigma_x^2(n) & \sigma_{xy}^2(n) \\ \sigma_{xy}^2(n) & \sigma_y^2(n) \end{bmatrix} \tag{3.19}$$

and is given by

$$R_n = A_2 R_0 A_2^T \tag{3.20}$$

where

$$R_0 = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & r^2\sigma_\theta^2 \end{bmatrix} \tag{3.21}$$

A_2 is defined in (3.4). σ_r^2 is the variance of the range measurement noise, σ_θ^2 is the variance of the bearing measurement noise, and r is the vehicle range.

3.3.3 Filtering Equations

The optimal estimates of the state vector after the measurement is given by

$$\hat{X}_n = \tilde{X}_n + K_n(Z_n - H\tilde{X}_n) \tag{3.22}$$

where $\tilde{X}_n$ the optimum estimate of the state vector before the measurement is given by (2.12). The Kalman gain matrix is given by (2.13). The predicted and filtered covariances are given by (2.14) and (2.15), respectively, where the quantities are as applicable to this model.

3.3.4 Steady State Results

For $\theta = 0$ (along the x axis), the tracker described above decouples into two independent one-dimensional trackers of the Friedland's model whose steady state gains and covariances are known. Hence, $\tilde{P}$, $\hat{P}$, and K matrices for the target at any bearing θ can be expressed in terms of those applicable for bearing $\theta = 0$. From (3.2) and (3.3),

$$\tilde{P} = A_{22}\tilde{P}_0 A_{22}^T \tag{3.23}$$

$$\hat{P} = A_{22}\hat{P}_0 A_{22}^T \tag{3.24}$$

$$K = A_{22} K_0 A_2^T \tag{3.25}$$

where A_{22} is given by (3.5).

3.3.5 Steady State Covariance Matrix $\tilde{P}_0$

$\tilde{P}_0$ can be written as a partitioned matrix

$$\tilde{P}_0 = \left[\begin{array}{c:c} \tilde{A} & \tilde{B} \\ \hdashline \tilde{B} & \tilde{C} \end{array}\right] \tag{3.26}$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are (2×2) diagonal matrices.

If $\tilde{A}_{kk}$, $\tilde{B}_{kk}$, and $\tilde{C}_{kk}$ (for $k = 1, 2$) are the unnormalized diagonal elements of $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$, then they are determined as follows:

1. Replace d by d_k and r by r_k in Eqs. (2.27).
2. Use them in (2.22) to determine $\tilde{P}_{11}$, $\tilde{P}_{12}$, and $\tilde{P}_{22}$, replacing σ_x by e_k.
3. Comparing (3.26) and (2.21), $\tilde{A}_{kk}$, $\tilde{B}_{kk}$, and $\tilde{C}_{kk}$ are obtained.

In this case, we get

$$\tilde{A}_{kk} = \left[\frac{d_k(d_k + 1)^2}{r_k^2}\right] e_k^2 \tag{3.27}$$

$$\tilde{B}_{kk} = \left[\frac{(d_k + 1)^2}{2r_k}\right] e_k \sigma_a T$$

$$\tilde{C}_{kk} = \left(\frac{d_k + 1}{2}\right) \sigma_a^2 T^2$$

where

$$d_k = \sqrt{1 + 2r_k} \tag{3.28}$$

$$r_k = 4e_k / \sigma_a T^2$$

For $k = 1, 2$, e_k is given by

$$e_1 = \sigma_r \tag{3.29}$$

$$e_2 = r\sigma_\theta \tag{3.30}$$

3.3.6 Steady State Covariance Matrix $\hat{P}_0$

The steady state filtered covariance matrix $\hat{P}_0$ may also be written as the partitioned matrix

$$\hat{P}_0 = \left[\begin{array}{c:c} \hat{A} & \hat{B} \\ \hdashline \hat{B} & \hat{C} \end{array}\right] \tag{3.31}$$

where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are 2×2 diagonal matrices.

If $\hat{A}_{kk}$, $\hat{B}_{kk}$, and $\hat{C}_{kk}$ (for $k = 1, 2$) are the unnormalized diagonal elements of $\hat{A}$, $\hat{B}$, and $\hat{C}$, then they are determined as follows:

1. Replace d by d_k and r by r_k in Eqs. (2.37).
2. Use them in (2.34) to determine $\hat{P}_{11}$, $\hat{P}_{12}$, and $\hat{P}_{22}$, replacing σ_x by e_k.
3. Comparing (3.31) and (2.33) and equating element to element, $\hat{A}_{kk}$, $\hat{B}_{kk}$, and $\hat{C}_{kk}$ are obtained as in the case of $\tilde{P}_0$ given by (3.27).

3.3.7 Steady State Gain Matrix K_0

K_0 may be partitioned in terms of two 2×2 diagonal matrices as

$$K_0 = \left[\begin{array}{c} G \\ \hdashline M \end{array}\right] \tag{3.32}$$

Comparing (3.32) with (2.28), the unnormalized diagonal elements G_{kk} and M_{kk} for $k = 1, 2$ may be expressed as in (2.30), replacing d by d_k and r by r_k.

3.3.8 Numerical Results

The steady state $\tilde{P}$, $\hat{P}$, and K matrices are evaluated from Eqs. (3.23) to (3.25) for the following values of the parameters. The results are presented below.

Parameters

$$\sigma_r = 0.16 \text{ nm}$$

$$\sigma_\theta = 0.23 \text{ degrees} = 0.0040143 \text{ rad}$$

$$T = 4 \text{ s}$$

$$r = 100 \text{ nm}$$

$$\theta = 0, 30 \text{ degrees}$$

$$\sigma_a = 0.01414 \text{ nm/s}^2$$

Computer Results

$$\tilde{P}_0 = \begin{bmatrix} .1058E+00 & .0000E+00 & .2050E-01 & .0000E+00 \\ .0000E+00 & .2992E+00 & .0000E+00 & .3837E-01 \\ .2050E-01 & .0000E+00 & .5727E-02 & .0000E+00 \\ .0000E+00 & .3837E-01 & .0000E+00 & .7835E-02 \end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} .1541E+00 & -.8373E-01 & .2497E-01 & -.7738E-02 \\ -.8373E-01 & .2508E+00 & -.7738E-02 & .3391E-02 \\ .2497E-01 & -.7738E-02 & .6254E-02 & -.9127E-03 \\ -.7738E-02 & .3391E-01 & -.9127E-03 & .7308E-02 \end{bmatrix}$$

$$\hat{P}_0 = \begin{bmatrix} .2061E-01 & .0000E+00 & .3994E-02 & .0000E+00 \\ .0000E+00 & .1047E+00 & .0000E+00 & .1343E-01 \\ .3994E-02 & .0000E+00 & .2528E-02 & .0000E+00 \\ .0000E+00 & .1343E-01 & .0000E+00 & .4635E-02 \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} .4164E-01 & -.3642E-01 & .6354E-02 & -.4087E-02 \\ -.3642E-01 & .8370E-01 & -.4087E-02 & .1107E-01 \\ .6354E-02 & -.4087E-02 & .3055E-02 & -.9127E-03 \\ -.4087E-02 & .1107E-01 & -.9127E-03 & .4109E-02 \end{bmatrix}$$

$$K_0 = \begin{bmatrix} .8052E+00 & .0000E+00 \\ .0000E+00 & .6499E+00 \\ .1560E+00 & .0000E+00 \\ .0000E+00 & .8336E-01 \end{bmatrix}$$

$$K = \begin{bmatrix} .7664E+00 & .6723E-01 \\ .6723E-01 & .6887E+00 \\ .1379E+00 & .3146E-01 \\ .3146E-01 & .1015E+00 \end{bmatrix}$$

When these matrices are evaluated by executing the Kalman filter matrix equations (2.17) to (2.19) for this model to steady state, we get nearly the same result. Comparing the values of $\tilde{P}$ and $\hat{P}$ matrices, it may be noted that the covariance goes down as a result of making an observation, even though the filter is in steady state.

3.4 TWO-DIMENSIONAL THREE-STATE FILTER: EXTENSION OF RAMACHANDRA'S MODEL I TO TWO DIMENSIONS

In this section, the one-dimensional model I of Ramachandra discussed in Chapter 2 is extended to two dimensions using the techniques and matrix transformation equations of Section 3.2. This two-dimensional model estimates the position, velocity, and acceleration of an aircraft moving with constant acceleration perturbed by a zero mean plant noise which accounts for maneuver and/or other random factors.

3.4.1 Dynamic Model

The equations of motion of the target are assumed to be described by the vector-matrix equation of the form (3.10) where

$$X_n^T = [x_n \quad y_n \quad \dot{x}_n \quad \dot{y}_n \quad \ddot{x}_n \quad \ddot{y}_n] \tag{3.33}$$

$$F = \begin{bmatrix} I & (T)I & (T^2/2)I \\ 0 & I & (T)I \\ 0 & 0 & I \end{bmatrix} \tag{3.34}$$

and

$$G = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \tag{3.35}$$

I denotes the 2×2 identity matrix, and 0 the 2×2 null matrix. F is a 6×6 transition matrix and G is a 6×2 input distribution matrix. a_n is the process noise perturbing the acceleration of the target with variance $Q = \sigma_a^2$ as described in Castella-Dunnebacke's model (Section 3.3.1).

3.4.2 Measurement Model

The range r and bearing θ of the target are assumed to be measured by a two-dimensional track-while-scan radar at uniform sampling intervals of time T seconds and all measurements are noisy. The measurement equation is the same as given by (3.14). The covariance matrix of measurement errors in cartesian coordinates is given by (3.20).

3.4.3 Filtering Equations

The filtering equations are given by (3.22) and (2.51) to (2.54) with quantities as applicable to this model.

3.4.4 Steady State Results

For $\theta = 0$ (along the x axis), this two-dimensional tracker decouples into two independent one-dimensional trackers of Ramachandra's model I whose properties are known. Hence the steady state covariance and gain matrices of the two-dimensional tracker may be expressed in terms of the known covariance and gain matrices of the one-dimensional tracker as

$$\tilde{P} = A_{23}\tilde{P}_0 A_{23}^T \tag{3.36}$$

$$\hat{P} = A_{23}\hat{P}_0 A_{23}^T \tag{3.37}$$

$$K = A_{23}K_0 A_2^T \tag{3.38}$$

where A_2 is defined in (3.4) and A_{23} in (3.9).

3.4.5 Steady State Covariance Matrix $\tilde{P}_0$

The $\tilde{P}_0$ matrix may be written as the partitioned matrix

$$\tilde{P}_0 = \left[\begin{array}{c:c:c} \tilde{A} & \tilde{B} & \tilde{C} \\ \hdashline \tilde{B} & \tilde{D} & \tilde{E} \\ \hdashline \tilde{C} & \tilde{E} & \tilde{F} \end{array}\right] \tag{3.39}$$

where the submatrices are all 2×2 diagonal matrices.

If $\tilde{A}_{kk}$, $\tilde{B}_{kk}$, $\tilde{C}_{kk}$, $\tilde{D}_{kk}$, $\tilde{E}_{kk}$, and $\tilde{F}_{kk}$ (for $k = 1, 2$) are the unnormalized diagonal elements of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$, and $\tilde{F}$, then they are determined as follows:

1. Replace m by m_k and r by r_k in Eqs. (2.70).
2. Use them in (2.61) to determine $\tilde{P}_{11}$, $\tilde{P}_{12}$, $\tilde{P}_{13}$, $\tilde{P}_{22}$, $\tilde{P}_{23}$, $\tilde{P}_{33}$, replacing σ_x by e_k.
3. Comparing (3.39) and (2.60) and equating element to element, $\tilde{P}_0$ is determined.

m_k is obtained by solving the cubic equation

$$r_k m_k^3 - 2[m_k^2 + 3m_k + 2] = 0 \tag{3.40}$$

3.4.6 Steady State Covariance Matrix $\hat{P}_0$

The $\hat{P}_0$ matrix may also be expressed as

$$\hat{P}_0 = \left[\begin{array}{c:c:c} \hat{A} & \hat{B} & \hat{C} \\ \hdashline \hat{B} & \hat{D} & \hat{E} \\ \hdashline \hat{C} & \hat{E} & \hat{F} \end{array}\right] \tag{3.41}$$

where the submatrices are all (2 × 2) diagonal matrices.

If $\hat{A}_{kk}$, $\hat{B}_{kk}$, $\hat{C}_{kk}$, $\hat{D}_{kk}$, $\hat{E}_{kk}$, and $\hat{F}_{kk}$ (for $k = 1, 2$) are the unnormalized diagonal elements of submatrices of $\hat{P}_0$, then they are determined as follows:

1. Replace m by m_k and r by r_k in Eqs. (2.80).
2. Use them in (2.77) to determine $\hat{P}_{11}$, $\hat{P}_{12}$, $\hat{P}_{13}$, $\hat{P}_{22}$, $\hat{P}_{23}$, $\hat{P}_{33}$, replacing σ_x by e_k.
3. Comparing (3.41) and (2.76), $\hat{P}_0$ is obtained.

3.4.7 Steady State Gain Matrix K_0

The gain matrix K_0 may be defined as

$$K_0 = \left[\begin{array}{c} G \\ \hdashline M \\ \hdashline N \end{array}\right] \tag{3.42}$$

where G, M, and N are 2×2 diagonal matrices. Comparing (3.42) with (2.72), the unnormalized diagonal elements G_{kk}, M_{kk}, and N_{kk} for $k = 1, 2$ may be obtained by replacing m by m_k and r by r_k in (2.75) and using them in (2.74) to determine the unnormalized elements.

The filter is initialized on the basis of three measurements.

3.4.8 Numerical Results

The steady state $\tilde{P}$, $\hat{P}$, and K matrices are evaluated from Eqs. (3.36) to (3.38) for the same values of parameters used in the numerical results of Castella-Dunnebacke's model (Section 3.3.8) and the results are presented below.

Computer Results

$$\tilde{P}_0 = \begin{bmatrix} .21E+00 & .00E+00 & .55E-01 & .00E+00 & .69E-02 & .00E+00 \\ .00E+00 & .68E+00 & .00E+00 & .13E+00 & .00E+00 & .13E-01 \\ .55E-01 & .00E+00 & .16E-01 & .00E+00 & .23E-02 & .00E+00 \\ .00E+00 & .13E+00 & .00E+00 & .32E-01 & .00E+00 & .36E-02 \\ .69E-02 & .00E+00 & .23E-02 & .00E+00 & .59E-03 & .00E+00 \\ .00E+00 & .13E-01 & .00E+00 & .36E-02 & .00E+00 & .71E-03 \end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} .33E+00 & -.20E+00 & .74E-01 & -.34E-01 & .84E-02 & -.26E-02 \\ -.20E+00 & .56E+00 & -.34E-01 & .11E+00 & -.26E-02 & .11E-01 \\ .74E-01 & -.34E-01 & .20E-01 & -.67E-02 & .27E-02 & -.57E-03 \\ -.34E-01 & .11E+00 & -.67E-02 & .28E-01 & -.57E-03 & .33E-02 \\ .84E-02 & -.26E-02 & .27E-02 & -.57E-03 & .62E-03 & -.51E-04 \\ -.26E-02 & .11E-01 & -.57E-03 & .33E-02 & -.51E-04 & .68E-03 \end{bmatrix}$$

$$\hat{P}_0 = \begin{bmatrix} .23E-01 & .00E+00 & .58E-02 & .00E+00 & .74E-03 & .00E+00 \\ .00E+00 & .13E+00 & .00E+00 & .25E-01 & .00E+00 & .25E-02 \\ .58E-02 & .00E+00 & -.25E-02 & .00E+00 & .76E-03 & .00E+00 \\ .00E+00 & .25E-01 & .00E+00 & .18E-03 & .00E+00 & .16E-02 \\ .74E-03 & .00E+00 & .76E-03 & .00E+00 & .39E-03 & .00E+00 \\ .00E+00 & .25E-02 & .00E+00 & .16E-02 & .00E+00 & .51E-03 \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} .50E-01 & -.46E-0 & .11E-01 & -.85E-02 & .12E-02 & -.76E-03 \\ -.46E-01 & .10E+00 & -.85E-02 & .21E-01 & -.76E-03 & .20E-02 \\ .11E-01 & -.85E-02 & -.18E-02 & -.11E-02 & .97E-03 & -.36E-03 \\ -.85E-02 & .21E-01 & -.11E-02 & -.48E-03 & -.36E-03 & .14E-02 \\ .12E-02 & -.76E-03 & .97E-03 & -.36E-03 & .42E-03 & -.51E-04 \\ -.76E-03 & .20E-02 & -.36E-03 & .14E-02 & -.51E-04 & .48E-03 \end{bmatrix}$$

$$K_0 = \begin{bmatrix} .89E+00 & .00E+00 \\ .00E+00 & .81E+00 \\ .23E+00 & .00E+00 \\ .00E+00 & .16E+00 \\ .29E-01 & .00E+00 \\ .00E+00 & .15E-01 \end{bmatrix}$$

$$K = \begin{bmatrix} .87E+00 & .37E-01 \\ .37E-01 & .83E+00 \\ .21E+00 & .30E-01 \\ .30E-01 & .18E+00 \\ .25E-01 & .58E-02 \\ .58E-02 & .19E-01 \end{bmatrix}$$

When these matrices are evaluated by executing the Kalman filter matrix equations (2.56) to (2.58) for this model, we get nearly the same result.

3.5 SUMMARY/SUGGESTED READING

The techniques and matrix transformation equations [1] for developing two-dimensional models for tracking in two-dimensional cartesian coordinates are given in Section 3.2. Using these techniques, the uncoupled one-dimensional trackers described in Chapter 1 may be extended to two dimensions for estimating position, velocity, and acceleration of an aircraft or similar vehicle.

In Section 3.3, Castella-Dunnebacke's model, which is an extension of Friedland's model to two dimensions for estimating position and velocity in the Cartesian coordinate system, is discussed and the steady state characteristics are expressed in compact forms using the techniques given in Section 3.2. In Section 3.4, a two-dimensional extension of Ramachandra's model I is given. This is also discussed in Ref. 3. The results of Ref. 3 will hold good only when the process noise along the x and y axes are of equal variance [1]. This restriction is eliminated in the two-dimensional three-state model described in Ref. 4. In Ref. 5, the one-dimensional model II of Ramachandra described in Section 2.4 is extended to two dimensions.

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4

Discrete-Time Three-Dimensional Tracking Filters

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4.1 INTRODUCTION

In this chapter, three-dimensional tracking filters are discussed for estimating the position, velocity, and also acceleration of an aircraft when the range r, bearing θ, and elevation φ of the target are measured at uniform sampling intervals of time T seconds through random noise by a three-dimensional radar sensor. The tracking operation is assumed to be performed in the cartesian coordinate system. The coupling between the quantities measured by the radar (r, θ, φ) and the cartesian coordinate system selected for tracking operation is explicitly considered in the development of the three dimensional models.

The steady state filter characteristics of the three dimensional trackers are analytically determined making use of the properties of the uncoupled one-dimensional trackers discussed in Chapter 2. These results are of practical interest in developing trackers for tracking aircraft and similar vehicles. These results also eliminate the real time execution of the complete filter equations, providing a significant saving in tracking and updating time.

4.2 TECHNIQUES AND MATRIX TRANSFORMATIONS

Fitzgerald [1] discusses the techniques and also the necessary matrix transformation equations for developing three-dimensional tracking filters. Using these techniques, we can express the steady state results of the three-dimensional filters in a concise form. If the covariances and gains are known in a system where there is no coupling, then the methods of transforming them for use in the coupled system are discussed in this section for three-dimensional two-state and three state filters separately.

4.2.1 Three-Dimensional Two-State Filters

Consider the case of tracking an aircraft or similar vehicle in the three-dimensional cartesian coordinate system with two state filters. Let the state vector be arranged as

$$X^T = [x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z}] \tag{4.1}$$

A three-dimensional track-while-scan radar sensor is assumed to measure the range r, bearing θ, and elevation φ of the vehicle at uniform sampling intervals of time T seconds and all measurements are noisy.

If P_0 and K_0 denote the covariance and Kalman gain matrices for tracking along the x axis corresponding to $\theta = 0$ and $\varphi = 0$, then for tracking at any arbitrary angles θ and φ, the covariance and gain matrices may be expressed as

$$P = A_{32} P_0 A_{32}^T \tag{4.2}$$

$$K = A_{32} K_0 A_3^T \tag{4.3}$$

where

$$A_{32} = \begin{bmatrix} A_3 & 0 \\ 0 & A_3 \end{bmatrix} \tag{4.4}$$

with

$$A_3 = \begin{bmatrix} \cos\theta\cos\varphi & \sin\theta & -\cos\theta\sin\varphi \\ \sin\theta\cos\varphi & -\cos\theta & -\sin\theta\sin\varphi \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \tag{4.5}$$

A_3 is the rotational matrix and A_{32} is a 6×6 matrix used for three-dimensional two-state filters. Equation (4.2) is applicable for both predicted and filtered covariance matrices $\tilde{P}$ and $\hat{P}$.

4.2.2 Three-Dimensional Three-State Filters

For tracking an aircraft or similar vehicle in three dimensions, the state vector is assumed to be arranged as

$$X^T = [x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z} \quad \ddot{x} \quad \ddot{y} \quad \ddot{z}] \tag{4.6}$$

If P_0 and K_0 are the covariance and gain matrices for tracking along the x axis corresponding to $\theta = 0$ and $\varphi = 0$, then for tracking at any arbitrary angles θ and φ, the covariance and gain matrices may be expressed as

$$P = A_{33} P_0 A_{33}^T \tag{4.7}$$

$$K = A_{33} K_0 A_3^T \tag{4.8}$$

where

$$A_{33} = \begin{bmatrix} A_3 & 0 & 0 \\ 0 & A_3 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \tag{4.9}$$

A_3 is the rotational matrix given by (4.5) and A_{33} is the 9×9 matrix used for three-dimensional three-state filters. Equation (4.7) is valid for both $\tilde{P}$ and $\hat{P}$ matrices. With (4.2) and (4.3) or (4.7) and (4.8), tracking is still nearly optimum if the rates of change of angles are slow [1].

4.3 RAMACHANDRA-SRINIVASAN'S MODEL: AN EXTENSION OF FRIEDLAND'S MODEL TO THREE DIMENSIONS

In this section, Ramachandra-Srinivasan's model [2], which is an extension of Friedland's model to three dimensions for estimating the position and velocity of an aircraft or similar vehicle, is discussed. The vehicle is assumed to be moving with a constant velocity motion perturbed by a zero mean random acceleration. The vehicle range r, bearing θ, and elevation φ are assumed to be measured by a three-dimensional track-while-scan radar at uniform sampling intervals of time T seconds and all measurements are noisy.

In this model, the coupling between the quantities measured by the radar and the cartesian xyz coordinate system selected for tracking operation is explicitly considered. The steady state characteristics of the filter are analytically determined under the assumption of a white noise maneuver model in three dimensions.

4.3.1 Dynamic Model

In three dimensions, the vehicle dynamics may be represented by the vector-matrix equation of the form (3.10) where the state vector is given by

$$X_n^T = [x_n \quad y_n \quad z_n \quad \dot{x}_n \quad \dot{y}_n \quad \dot{z}_n] \tag{4.10}$$

F and G are also of the same form given by (3.12) and (3.13), where I is a 3×3 identity matrix and 0 is a 3×3 null matrix. F is a 6×6 matrix and G is a 6×3 matrix. a_n is the random acceleration acting on the vehicle with zero mean and constant variance $Q = \sigma_a^2$ and is assumed to be of equal variance and also independent along the x, y, and z axes. Acceleration values at different scans are assumed to be uncorrelated (white noise maneuver nodel).

4.3.2 Measurement Model

The measurement equation may be written as (3.14) where

$$Z_n = \begin{bmatrix} x_m(n) \\ y_m(n) \\ z_m(n) \end{bmatrix} \tag{4.11}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{4.12}$$

and

$$V_n = \begin{bmatrix} v_x(n) \\ v_y(n) \\ v_z(n) \end{bmatrix} \tag{4.13}$$

$x_m(n)$ = measured x coordinate at scan n
$y_m(n)$ = measured y coordinate at scan n
$z_m(n)$ = measured z coordinate at scan n
$v_x(n)$ = random noise on x measurement at scan n
$v_y(n)$ = random noise on y measurement at scan n
$v_z(n)$ = random noise on z measurement at scan n

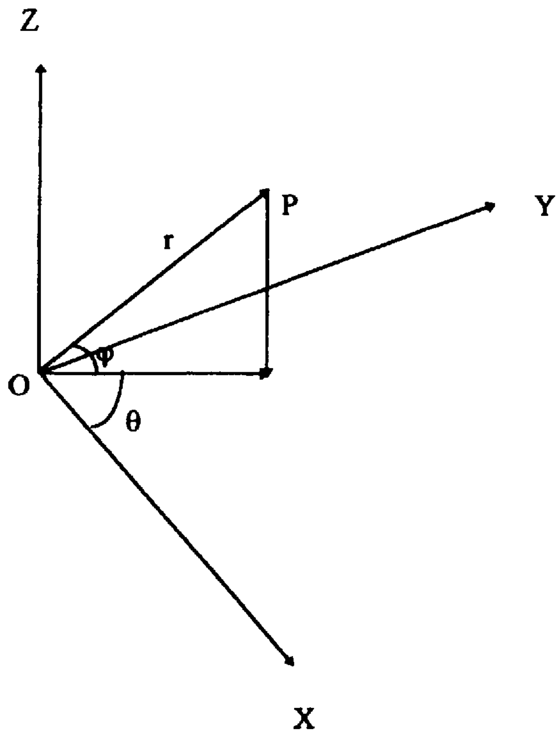

Figure 4.1 Three-dimensional tracking geometry.

Let the target range r, bearing θ, and elevation φ be measured by a three-dimensional radar sensor.

From the tracking geometry illustrated in Figure 4.1,

$$x_n = r(n)\cos\theta(n)\cos\varphi(n) \tag{4.14}$$
$$y_n = r(n)\sin\theta(n)\cos\varphi(n)$$
$$z_n = r(n)\sin\varphi(n)$$

As the measurements are in polar coordinates and tracking is done in cartesian coordinates, the measurements are coupled. The covariance matrix of the measurement errors in cartesian coordinates will be of the form

$$R_n = \begin{bmatrix} \sigma_x^2(n) & \sigma_{xy}^2(n) & \sigma_{xz}^2(n) \\ \sigma_{xy}^2(n) & \sigma_y^2(n) & \sigma_{yz}^2(n) \\ \sigma_{xz}^2(n) & \sigma_{yz}^2(n) & \sigma_z^2(n) \end{bmatrix} \tag{4.15}$$

and is given by

$$R_n = A_3 R_0 A_3^T \tag{4.16}$$

where

$$R_0 = \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & r^2\sigma_\theta^2 & 0 \\ 0 & 0 & r^2\sigma_\varphi^2 \end{bmatrix} \tag{4.17}$$

4.3.3 Filtering Equations

The filtering equations are given by (3.22) and (2.12) to (2.15) with quantities as applicable to this model.

4.3.4 Steady State Results

For $\theta = \varphi = 0$ (along the x axis), the tracker described so far degenerates to three independent one-dimensional trackers of Friedland's model whose characteristics are known. Hence the steady state covariance and gain matrices are given by

$$\tilde{P} = A_{32}\tilde{P}_0 A_{32}^T \tag{4.18}$$

$$\hat{P} = A_{32}\hat{P}_0 A_{32}^T \tag{4.19}$$

$$K = A_{32} K_0 A_3^T \tag{4.20}$$

$\tilde{P}_0$ is the partitioned matrix of the form given by (3.26) where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are 3×3 diagonal matrices with their diagonal elements given by (3.27) for $k = 1, 2, 3$.

For $k = 1, 2$, the values of e_k are given in (3.29) and (3.30). For $k = 3$,

$$e_3 = r\sigma_\varphi$$

$\hat{P}_0$ is of the form given by (3.31) where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are 3×3 diagonal matrices with their diagonal elements determined as in the case of the two-dimensional model for $k = 1, 2, 3$.

K_0 may be expressed as the partitioned matrix of the form (3.32), where G and M are 3×3 diagonal matrices. The diagonal elements of these submatrices are determined in the same way as in the two-dimensional model for $k = 1, 2, 3$.

4.3.5 Numerical Results

The steady state $\tilde{P}$, $\hat{P}$, and K matrices are evaluated from Eqs. (4.18) to (4.20) for the values of parameters used in the numerical results of Castella-Dunnebacke's model (Section 3.3.8) along with

$$\sigma_{\varphi} = 1 \text{ degree} = 0.017450 \text{ rad}$$

$$\varphi = 0.30 \text{ degrees}$$

and the results are presented below:

$$\tilde{P}_0 = \begin{bmatrix} .11E{+}00 & .00E{+}00 & .00E{+}00 & .21E{-}01 & .00E{+}00 & .00E{+}00 \\ .00E{+}00 & .30E{+}00 & .00E{+}00 & .00E{+}00 & .38E{-}01 & .00E{+}00 \\ .00E{+}00 & .00E{+}00 & .20E{+}01 & .00E{+}00 & .00E{+}00 & .13E{+}00 \\ .21E{-}01 & .00E{+}00 & .00E{+}00 & .57E{-}02 & .00E{+}00 & .00E{+}00 \\ .00E{+}00 & .38E{-}01 & .00E{+}00 & .00E{+}00 & .78E{-}02 & .00E{+}00 \\ .00E{+}00 & .00E{+}00 & .13E{+}00 & .00E{+}00 & .00E{+}00 & .14E{-}01 \end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} .51E{+}00 & .12E{+}00 & -.72E{+}00 & .45E{-}01 & .38E{-}02 & -.40E{-}01 \\ .12E{+}00 & .37E{+}00 & -.41E{+}00 & .38E{-}02 & .41E{-}01 & -.23E{-}01 \\ -.72E{+}00 & -.41E{+}00 & .15E{+}01 & -.40E{-}01 & -.23E{-}01 & .10E{+}00 \\ .45E{-}01 & .38E{-}02 & -.40E{-}01 & .79E{-}02 & .12E{-}04 & -.32E{-}02 \\ .38E{-}02 & .41E{-}01 & -.23E{-}01 & .12E{-}04 & .78E{-}02 & -.18E{-}02 \\ -.40E{-}01 & -.23E{-}01 & .10E{+}00 & -.32E{-}02 & -.18E{-}02 & .12E{-}01 \end{bmatrix}$$

$$\hat{P}_0 = \begin{bmatrix} .21E{-}01 & .00E{+}00 & .00E{+}00 & .40E{-}02 & .00E{+}00 & .00E{+}00 \\ .00E{+}00 & .10E{+}00 & .00E{+}00 & .00E{+}00 & .13E{-}01 & .00E{+}00 \\ .00E{+}00 & .00E{+}00 & .12E{+}01 & .00E{+}00 & .00E{+}00 & .77E{-}01 \\ .40E{-}02 & .00E{+}00 & .00E{+}00 & .25E{-}02 & .00E{+}00 & .00E{+}00 \\ .00E{+}00 & .13E{-}01 & .00E{+}00 & .00E{+}00 & .46E{-}02 & .00E{+}00 \\ .00E{+}00 & .00E{+}00 & .77E{-}01 & .00E{+}00 & .00E{+}00 & .11E{-}01 \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} .27E{+}00 & .93E{-}01 & -.45E{+}00 & .20E{-}01 & .38E{-}02 & -.27E{-}01 \\ .93E{-}01 & .16E{+}00 & -.26E{+}00 & .38E{-}02 & .16E{-}01 & -.16E{-}01 \\ -.45E{+}00 & -.26E{+}00 & .91E{+}00 & -.27E{-}01 & -.16E{-}01 & .58E{-}01 \\ .20E{-}01 & .38E{-}02 & -.27E{-}01 & .47E{-}02 & .12E{-}04 & -.32E{-}02 \\ .38E{-}02 & .16E{-}01 & -.16E{-}01 & .12E{-}04 & .46E{-}02 & -.18E{-}02 \\ -.27E{-}01 & -.16E{-}01 & .58E{-}01 & -.32E{-}02 & -.18E{-}02 & .89E{-}02 \end{bmatrix}$$

$$K_0 = \begin{bmatrix} .81E{+}00 & .00E{+}00 & .00E{+}00 \\ .00E{+}00 & .65E{+}00 & .00E{+}00 \\ .00E{+}00 & .00E{+}00 & .40E{+}00 \\ .16E{+}00 & .00E{+}00 & .00E{+}00 \\ .00E{+}00 & .83E{-}01 & .00E{+}00 \\ .00E{+}00 & .00E{+}00 & .25E{-}01 \end{bmatrix}$$

$$K = \begin{bmatrix} .69E+00 & .23E-01 & .15E+00 \\ .23E-01 & .66E+00 & .88E-01 \\ .15E+00 & .88E-01 & .50E+00 \\ .11E+00 & .17E-01 & .49E-01 \\ .17E-01 & .93E-01 & .28E-01 \\ .49E-01 & .28E-01 & .58E-01 \end{bmatrix}$$

When these matrices are evaluated by executing the Kalman filter matrix equations (2.17) to (2.19) for this model to steady state, we get nearly the same results. It may be observed that the covariance goes down as a result of making an observation, even though the filter is in steady state.

4.4 EXTENSION OF RAMACHANDRA'S MODEL II TO THREE DIMENSIONS

In this section, the one-dimensional model II of Ramachandra is extended to three dimensions using the techniques and matrix transformation equations of Section 4.2, and a three-dimensional tracker is developed. This tracker estimates the position, velocity, and acceleration of an aircraft moving with constant acceleration and is acted upon by a zero mean random rate of change of acceleration which accounts for maneuvers and/or other random factors.

4.4.1 Dynamic Model

In three-dimensional cartesian coordinate system, the equations of motion of the target are assumed to be described by the vector-matrix equation of the form (3.10), where X_n is the vehicle state vector consisting of position, velocity, and acceleration components and is a nine-element vector defined as

$$X_n^T = [x_n \quad y_n \quad z_n \quad \dot{x}_n \quad \dot{y}_n \quad \dot{z}_n \quad \ddot{x}_n \quad \ddot{y}_n \quad \ddot{z}_n] \tag{4.22}$$

F is the transition matrix of dimension 9×9 and is given by

$$F = \begin{bmatrix} I & (T)I & (T^2/2)I \\ 0 & I & (T)I \\ 0 & 0 & I \end{bmatrix} \tag{4.23}$$

G is the input distribution matrix of dimension 9×3 and is given by

$$G = \begin{bmatrix} (T^3/6)I \\ (T^2/2)I \\ (T)I \end{bmatrix} \tag{4.24}$$

I denotes the 3×3 identity matrix. a_n is the rate of change of acceleration assumed to be a random constant between successive scans with zero mean and constant variance σ_a^2. It is assumed to be of equal variance and also independent along x, y, and z axes. The values of a_n at different scans are assumed to be uncorrelated (white noise maneuver model).

4.4.2 Measurement Model

The range r, bearing θ, and elevation φ of the target are assumed to be measured by a three-dimensional track-while-scan radar at uniform sampling intervals of time T seconds and all measurements are assumed to be corrupted with range noise and angular noise. The tracking geometry is illustrated in Figure 4.1. The measurement equation is of the form given by (3.14), where Z_n, H, and V_n are as defined in (4.11) to (4.13). The covariance matrix of measurement noise is given by (4.16).

4.4.3 Filtering Equations

The optimal estimates of state vector after the measurement is given by (3.22), and before the measurement by (2.12). The Kalman gain matrix is given by (2.13). The predicted and filtered covariance are given by (2.14) and (2.15), respectively.

4.4.4 Steady State Results

For $\theta = \varphi = 0$ (along the x axis), this three-dimensional tracker decouples into three one-dimensional trackers of Ramachandra's model II whose steady state covariances and gains are known. Using the matrix transformation equations (4.7) and (4.8), the steady state covariance and gain matrices of the three dimensional tracker are given by

$$\tilde{P} = A_{33}\tilde{P}_0 A_{33}^T \tag{4.25}$$

$$\hat{P} = A_{33}\hat{P}_0 A_{33}^T \tag{4.26}$$

$$K = A_{33}K_0 A_3^T \tag{4.27}$$

The $\tilde{P}_0$ matrix is as defined by (3.39), and its submatrices are all 3×3 diagonal matrices whose diagonal elements are determined as follows:

1. Replace S by S_k and r by r_k in Eqs. (2.95).
2. Use them in (2.85) to determine the unnormalized covariances, replacing σ_x by e_k.

3. Comparing (3.39) and (2.60) and equating element to element, $\tilde{P}_0$ is obtained where

$$r_k = \frac{12e_k}{\sigma_a T^3} \tag{4.28}$$

For $k = 1, 2, 3$, e_1 and e_2 are as given in (3.29) and (3.30). e_3 is given in (4.21). S_k is obtained by solving the biquadratic equation

$$S_k^4 - 6S_k^3 + 10S_k^2 - 6mS_k + n = 0 \tag{4.29}$$

for $k = 1, 2, 3$.

The $\hat{P}_0$ matrix may also be expressed as (3.41) where the submatrices are all 3×3 diagonal matrices whose diagonal elements are determined as follows:

1. Replace S by S_k and r by r_k in Eqs. (2.101).
2. Use them in (2.85), which is equally applicable for filtered covariances also (replacing tildes by hats on both sides), to determine the unnormalized filtered covariances, replacing σ_x by e_k.
3. Comparing (3.41) and (2.76) and equating element to element, $\hat{P}_0$ is obtained.

The K_0 matrix is given by (3.42) as a partitioned matrix and its submatrices are determined as follows:

1. Replace S by S_k and r by r_k in Eqs. (2.100).
2. Use them in (2.74), which is equally applicable for this model, to determine the unnormalized gains.
3. Comparing (3.42) and (2.72) and equating element to element, K_0 is obtained.

The filter is initialized on the basis of the first three measurements.

4.4.5 Numerical Results

The steady state $\tilde{P}$, $\hat{P}$, and K matrices are evaluated from Eqs. (4.25) to (4.27) for the values of parameters used in the numerical results of Section 4.3.5, and the results are tabulated.

$$\tilde{P}_0 = \begin{bmatrix}
.83E+00 & .00E+00 & .00E+00 & .29E+00 & .00E+00 & .00E+00 & .52E-01 & .00E+00 & .00E+00 \\
.00E+00 & .20E+01 & .00E+00 & .00E+00 & .57E+00 & .00E+00 & .00E+00 & .83E-01 & .00E+00 \\
.00E+00 & .00E+00 & .12E+02 & .00E+00 & .00E+00 & .23E+01 & .00E+00 & .00E+00 & .22E+00 \\
.29E+00 & .00E+00 & .00E+00 & .11E+00 & .00E+00 & .00E+00 & .23E-01 & .00E+00 & .00E+00 \\
.00E+00 & .57E+00 & .00E+00 & .00E+00 & .18E+00 & .00E+00 & .00E+00 & .31E-01 & .00E+00 \\
.00E+00 & .00E+00 & .23E+01 & .00E+00 & .00E+00 & .53E+00 & .00E+00 & .00E+00 & .62E-01 \\
.52E-01 & .00E+00 & .00E+00 & 23E-01 & .00E+00 & .00E+00 & .60E-02 & .00E+00 & .00E+00 \\
.00E+00 & .83E-01 & .00E+00 & .00E+00 & .31E-01 & .00E+00 & .00E+00 & .71E-02 & .00E+00 \\
.00E+00 & .00E+00 & .22E+00 & .00E+00 & .00E+00 & .62E-01 & .00E+00 & .00E+00 & .99E-02
\end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} .32E+01 & .71E+00 & -.42E+01 & .73E+00 & .95E-01 & -.75E+00 & .91E-01 & .47E-02 & -.63E-01 \\ .71E+00 & .24E+01 & -.24E+01 & .95E-01 & .62E+00 & -.43E+00 & .47E-02 & .86E-01 & -.36E-01 \\ -.42E+01 & -.24E+01 & .92E+01 & -.75E+00 & -.43E+00 & .18E+01 & -.63E-01 & -.36E-01 & .18E+00 \\ .73E+00 & .95E-01 & -.75E+00 & .21E+00 & .13E-01 & -.16E+00 & .32E-01 & .46E-03 & -.15E-01 \\ .95E-01 & .62E+00 & -.43E+00 & .13E-01 & .19E+00 & -.91E-01 & .46E-03 & .32E-01 & -.84E-02 \\ -.75E+00 & -.43E+00 & .18E+01 & -.16E+00 & -.91E-01 & .43E+00 & -15E-01 & -.84E-02 & .52E-01 \\ .91E-01 & .47E-02 & -.63E-01 & .32E-01 & .46E-03 & -.15E-01 & 70E-02 & -.36E-04 & -.15E-02 \\ .47E-02 & .86E-01 & -.36E-01 & .46E-03 & .32E-01 & -.84E-02 & -.36E-04 & .70E-02 & -.85E-03 \\ -.63E-01 & -.36E-01 & .18E+00 & -.15E-01 & -.84E-02 & .52E-01 & -.15E-02 & -.85E-03 & .89E-02 \end{bmatrix}$$

$$\hat{P}_0 = \begin{bmatrix} .25E-01 & .00E+00 & .00E+00 & .86E-02 & .00E+00 & .00E+00 & .16E-02 & .00E+00 & .00E+00 \\ .00E+00 & .15E+00 & .00E+00 & .00E+00 & .42E-01 & .00E+00 & .00E+00 & .62E-02 & .00E+00 \\ .00E+00 & .00E+00 & .24E+01 & .00E+00 & .00E+00 & .46E+00 & .00E+00 & .00E+00 & .44E-01 \\ .86E-02 & .00E+00 & .00E+00 & .12E-01 & .00E+00 & .00E+00 & .49E-02 & .00E+00 & .00E+00 \\ .00E+00 & .42E-01 & .00E+00 & .00E+00 & .34E-01 & .00E+00 & .00E+00 & .94E-02 & .00E+00 \\ .00E+00 & .00E+00 & .46E+00 & .00E+00 & .00E+00 & .18E+00 & .00E+00 & .00E+00 & .28E-01 \\ .16E-02 & .00E+00 & .00E+00 & .49E-02 & .00E+00 & .00E+00 & .28E-02 & .00E+00 & .00E+00 \\ .00E+00 & .62E-02 & .00E+00 & .00E+00 & .94E-02 & .00E+00 & .00E+00 & .39E-02 & .00E+00 \\ .00E+00 & .00E+00 & .44E-01 & .00E+00 & .00E+00 & .28E-01 & .00E+00 & .00E+00 & .67E-02 \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} .51E+00 & .21E+00 & -.90E+00 & .10E+00 & .34E-01 & -.17E+00 & .11E-01 & .26E-02 & -.16E-01 \\ .21E+00 & .27E+00 & -.52E+00 & .34E-01 & .62E-01 & -.98E-01 & .26E-02 & .77E-02 & -.93E-02 \\ -.90E+00 & -.52E+00 & .18E+01 & -.17E+00 & -.98E-01 & .35E+00 & -.16E-01 & -.93E-02 & .34E-01 \\ .10E+00 & .34E-01 & -.17E+00 & .50E-01 & .91E-02 & -.65E-01 & 10E-01 & .60E-03 & -.87E-02 \\ .34E-01 & .62E-01 & -.98E-01 & .91E-02 & .39E-01 & -.37E-01 & .60E-03 & .97E-02 & -.50E-02 \\ -.17E+00 & -.98E-01 & .35E+00 & -.65E-01 & -.37E-01 & .14E+00 & -.87E-02 & -.50E-02 & .22E-01 \\ .11E-01 & .26E-02 & -.16E-01 & .10E-01 & .60E-03 & -.87E-02 & .38E-02 & -.36E-04 & -.15E-02 \\ .26E-02 & .77E-02 & -.93E-02 & .60E-03 & .97E-02 & -.50E-02 & -.36E-04 & .39E-02 & -.85E-03 \\ -.16E-01 & -.93E-02 & .34E-01 & -.87E-02 & -.50E-02 & .22E-01 & -.15E-02 & -.85E-03 & .57E-02 \end{bmatrix}$$

$$K_0 = \begin{bmatrix} .97E+00 & .00E+00 & .00E+00 \\ .00E+00 & .93E+00 & .00E+00 \\ .00E+00 & .00E+00 & .80E+00 \\ .34E+00 & .00E+00 & .00E+00 \\ .00E+00 & .26E+00 & .00E+00 \\ .00E+00 & .00E+00 & .15E+00 \\ .61E-01 & .00E+00 & .00E+00 \\ .00E+00 & .38E-01 & .00E+00 \\ .00E+00 & .00E+00 & .15E-01 \end{bmatrix}$$

$$K = \begin{bmatrix} .93E+00 & .69E-03 & .64E-01 \\ .69E-03 & .93E+00 & .37E-01 \\ .64E-01 & .37E-01 & .84E+00 \\ .28E+00 & .12E-01 & .70E-01 \\ .12E-01 & .27E+00 & .40E-01 \\ .70E-01 & .40E-01 & .20E+00 \\ .47E-01 & .48E-02 & .17E-01 \\ .48E-02 & .41E-01 & .10E-01 \\ .17E-01 & .10E-01 & .26E-01 \end{bmatrix}$$

When these matrices are evaluated by executing the Kalman filter matrix equations (2.17) to (2.19) to steady state with quantities as applicable to this model, it may be verified that we get nearly the same result.

4.5 SUMMARY/SUGGESTED READING

The techniques and necessary matrix transformation equations for developing three-dimensional models for tracking in cartesian coordinates are given in Section 4.2. Using these techniques, the uncoupled one dimensional trackers described in Chapter 2 may be extended to three dimensions for estimating position, velocity, and acceleration of an aircraft or similar vehicles. In Section 4.3, the one dimensional Friedland's model is extended to three dimensions and the steady state characteristics are expressed in compact forms using the techniques given in Section 4.2. The covariance and Kalman gain matrices are expressed in terms of those matrices which are applicable for tracking along the x axis. In Refs. 2 and 3, this model is discussed and the steady state results are given in scalar forms. In Ref. 4, Ramachandra's model II is extended to three dimensions and the steady state results are analytically determined.

In extending the uncoupled models to higher dimensions for tracking in cartesian coordinate system, the following two assumptions have been made.

1. The maneuvers along the x, y, and z axes are independent.
2. The maneuver noise is of equal variance along the x, y, and z axes.

These assumptions are eliminated in the alternate maneuver model discussed in Ref. 5. There it is assumed that both the maneuver characteristics and the measurement uncertainties are known in polar coordinates. These are coupled to the cartesian coordinate system, explicitly assuming that the axes of the plant noise ellipsoid and the measurement noise ellipsoid are parallel. The covariance and Kalman gain matrices are expressed in terms of those matrices which are applicable for tracking in polar coordinates.

In Ref. 6, Baheti presents an efficient approximation of the Kalman filter for target tracking. The filter gains and the tracking errors of the approximate method are shown to be identical to the extended Kalman filter with reduced computation requirements.

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Continuous-Time One-Dimensional Tracking Filters with Position Measurements

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5.1 INTRODUCTION

The continuous-time exponentially correlated velocity and acceleration (ECV and ECA) models of Fitzgerald [1] are presented in this chapter for continuous position measurements. Their steady state solutions are also discussed.

5.2 FITZGERALD'S CONTINUOUS-TIME ECV TARGET TRACKING FILTER

Consider a continuous-time one-dimensional two-state Kalman-Bucy filter for tracking a vehicle such as an aircraft moving with an exponentially correlated velocity (ECV) perturbed by a white noise process of spectral density q. The position of the target is assumed to be measured continuously with a white measurement noise of spectral density r_0.

5.2.1 Dynamic Model

The ECV tracking model [1] is described by the equations of motion given by

$$\dot{X} = FX + W \tag{5.1}$$

where

$$F = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix} \tag{5.2}$$

$$X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{5.3}$$

and

$$W = \begin{bmatrix} 0 \\ \omega \end{bmatrix} \tag{5.4}$$

X is the state vector consisting of the target position x and target velocity $\dot{x}$ at time t, and W is a white noise vector with covariance Q given by

$$E\{W(t)W^T(n)\} = Q\delta(t-n) \tag{5.5}$$

The covariance matrix Q is given by

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} \tag{5.6}$$

q is the spectral density of the white noise process ω given by

$$q = \frac{2\sigma_v^2}{\tau} \tag{5.7}$$

where σ_v is the standard deviation of the target velocity.

The resulting $\dot{x}$ process is exponentially correlated with correlation time τ and variance $\sigma_v^2 = q\tau/2$.

5.2.2 Measurement Model

The position of the target is assumed to be measured continuously. The measurement equation is given by

$$x_m = HX + v \tag{5.8}$$

where

$$H = [1 \quad 0]$$

v is a white measurement noise of spectral density $R = r_0$.

5.2.3 Covariance Matrix

The covariance matrix is given by

$$\dot{P} = FP + PF^T - PH^T R^{-1} HP + Q \tag{5.9}$$

5.2.4 Kalman Gain Matrix

The gain matrix is given by

$$K = PH^T R^{-1} \tag{5.10}$$

5.2.5 Steady State Covariance Matrix

A closed-form steady state solution for the Kalman filter covariance matrix is analytically obtained in [1] by directly solving the algebraic Riccati equation (5.9).

Let the error covariance matrix P and its derivative $\dot{P}$ be defined as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \tag{5.11}$$

$$\dot{P} = \begin{bmatrix} \dot{P}_{11} & \dot{P}_{12} \\ \dot{P}_{12} & \dot{P}_{22} \end{bmatrix} \tag{5.12}$$

This satisfies the differential equation (5.9) which is equivalent to the following three scalar differential equations:

$$\dot{P}_{11} = 2P_{12} - P_{11}^2/r_0 \tag{5.13}$$

$$\dot{P}_{12} = P_{22} - P_{12}/\tau - P_{11}P_{12}/r_0 \tag{5.14}$$

$$\dot{P}_{22} = q - 2P_{22}/\tau - P_{12}^2/r_0 \tag{5.15}$$

Let the normalized covariances be defined as

$$Y_{11} = P_{11}/(r_0/\tau) \tag{5.16}$$
$$Y_{12} = P_{12}/(r_0/\tau^2)$$
$$Y_{22} = P_{22}/(r_0/\tau^3)$$

The steady state solution is one which drives all the derivatives in (5.13) to (5.15) to zero. Thus, in the steady state we have from (5.13) to (5.15) and (5.16),

$$Y_{11}^2 = 2Y_{12} \tag{5.17}$$
$$Y_{22} = Y_{12}(1 + Y_{11}) \tag{5.18}$$
$$2Y_{22} + Y_{12}^2 = r \tag{5.19}$$

where

$$r = q\tau^4/r_0 \tag{5.20}$$

Let

$$Y_{11} = a \tag{5.21}$$

Then from (5.17),

$$Y_{12} = a^2/2 \tag{5.22}$$

and from (5.18) and (5.22), we get

$$Y_{22} = a^2(1 + a)/2 \tag{5.23}$$

Putting (5.22) and (5.23) in (5.19) yields

$$a^2(a + 2)^2 = 4r$$

or

$$a(a + 2) = 2\sqrt{r}$$

or

$$a = -1 + \sqrt{1 + 2\sqrt{r}} \tag{5.24}$$

and hence Y_{11}, Y_{12}, and Y_{13} are obtained from (5.21) to (5.23).

5.2.6 Steady State Gains

The gain given by (5.10) is equivalent to

$$K_1 = \frac{P_{11}}{r_0} \tag{5.25}$$

$$K_2 = \frac{P_{12}}{r_0} \tag{5.26}$$

5.3 RANDOM WALK VELOCITY MODEL

As $\tau \to \infty$, the ECV model reduces to a random walk velocity (RWV) or white acceleration model. By equating the derivatives to zero and letting $\tau \to \infty$ in (5.13) to (5.15), the steady state solution for this case becomes

$$P_{11}/r_0 = \sqrt{2}(q/r_0)^{1/4} \tag{5.27}$$

$$P_{12}/r_0 = (q/r_0)^{1/2} \tag{5.28}$$

$$P_{22}/r_0 = \sqrt{2}(q/r_0)^{3/4} \tag{5.29}$$

5.4 NASH'S GENERAL SOLUTION TO ECV FILTER

The steady state solution obtained by Nash [2] is given by

$$P_{11}/(r\lambda) = K_1/\lambda = \sqrt{1+2\alpha} - 1 \tag{5.30}$$

$$P_{12}/(r\lambda^2) = K_2/\lambda^2 = \alpha + 1 - \sqrt{1+2\alpha} \tag{5.31}$$

$$P_{22}/(r\lambda^3) = (1+\alpha)(\sqrt{1+2\alpha} - 1) - \alpha \tag{5.32}$$

where

$$\alpha = (1/\lambda^2)\sqrt{q/r_0} \tag{5.33}$$

and

$$\lambda = 1/\tau \tag{5.34}$$

It may be easily seen that the steady state solutions of Fitzgerald and Nash are identical.

5.5 FITZGERALD'S CONTINUOUS-TIME ECA TARGET TRACKING FILTER

Consider a continuous-time one-dimensional three-state Kalman-Bucy filter for tracking a vehicle such as an aircraft moving with an exponentially correlated acceleration (ECA) perturbed by a white noise process of spectral density q. The position of the target is assumed to be measured continuously with a white measurement noise of spectral density r_0.

5.5.1 Dynamic Model

The ECA tracking filter [1] is described by equations of motion of the form given by (5.1) with

$$X = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} \tag{5.35}$$

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau \end{bmatrix} \tag{5.36}$$

$$W = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \tag{5.37}$$

X is the state vector consisting of the target position x, target velocity $\dot{x}$, and target acceleration $\ddot{x}$ at time t. W is a white noise vector which satisfies (5.5). The covariance matrix Q is given by

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{bmatrix} \tag{5.38}$$

where q is the spectral density of the white noise process ω given by

$$q = 2\sigma_a^2/\tau \tag{5.39}$$

σ_a is the standard deviation of the target acceleration.

The resulting $\ddot{x}$ process is exponentially correlated with correlation time τ and variance $\sigma_a^2 = q\tau/2$.

5.5.2 Measurement Model

The position of the target is assumed to be measured continuously with a white measurement noise υ of spectral density

$$R = r_0 \tag{5.40}$$

The measurement equation is of the form given by (5.8) with

$$H = [1 \quad 0 \quad 0] \tag{5.41}$$

5.5.3 Covariance Matrix

The covariance matrix satisfies the differential equation (5.9).

5.5.4 Kalman Gain Matrix

The Kalman gain matrix is given by (5.10).

5.5.5 Steady State Covariance Matrix

A closed-form steady state solution for the covariance matrix is analytically obtained by directly solving the algebraic Riccati equation (5.9).

Let the error covariance matrix P and its derivative $\dot{P}$ be defined as

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} \tag{5.42}$$

and

$$\dot{P} = \begin{bmatrix} \dot{P}_{11} & \dot{P}_{12} & \dot{P}_{13} \\ \dot{P}_{12} & \dot{P}_{22} & \dot{P}_{23} \\ \dot{P}_{13} & \dot{P}_{23} & \dot{P}_{33} \end{bmatrix} \tag{5.43}$$

Let the normalized covariances be defined as

$$\begin{aligned} Y_{11} &= P_{11}/(r_0/\tau) \\ Y_{12} &= P_{12}/(r_0/\tau^2) \\ Y_{13} &= P_{13}/(r_0/\tau^3) \\ Y_{22} &= P_{22}/(r_0/\tau^3) \\ Y_{23} &= P_{23}/(r_0/\tau^4) \\ Y_{33} &= P_{33}/(r_0/\tau^5) \end{aligned} \tag{5.44}$$

Equation (5.9) is equivalent to the following six scalar differential equations:

$$\dot{P}_{11} = 2P_{12} - P_{11}^2/r_0 \tag{5.45}$$

$$\dot{P}_{22} = 2P_{23} - P_{12}^2/r_0 \tag{5.46}$$

$$\dot{P}_{13} = P_{23} - P_{13}/\tau - P_{11}P_{13}/r_0 \tag{5.47}$$

$$\dot{P}_{12} = P_{22} + P_{13} - P_{11}P_{12}/r_0 \tag{5.48}$$

$$\dot{P}_{33} = -2P_{33}/\tau - P_{13}^2/r_0 + q \tag{5.49}$$

$$\dot{P}_{23} = P_{33} - P_{23}/\tau - P_{12}P_{13}/r_0 \tag{5.50}$$

The steady state solution is the one which drives all the derivatives to zero. Thus in the steady state, we have from (5.44) and (5.45) to (5.50),

$$Y_{12} = Y_{11}^2/2 \tag{5.51}$$

$$Y_{23} = Y_{12}^2/2 \tag{5.52}$$

$$Y_{13} = Y_{23}/(1 + Y_{11}) \tag{5.53}$$

$$Y_{22} = Y_{11}Y_{12} - Y_{13} \tag{5.54}$$

$$Y_{33} = r/64 - Y_{13}^2/2 \tag{5.55}$$

$$Y_{33} = Y_{23} + Y_{12}Y_{13} \tag{5.56}$$

where

$$r = 32q\tau^6/r_0 \tag{5.57}$$

Let

$$Y_{11} = a \tag{5.58}$$

Then from (5.51) to (5.55), we have

$$Y_{12} = a^2/2 \tag{5.59}$$

$$Y_{23} = a^4/8 \tag{5.60}$$

$$Y_{13} = a^4/[8(1 + a)] \tag{5.61}$$

$$Y_{22} = a^3(3a + 4)/[8(1 + a)] \tag{5.62}$$

$$Y_{33} = \frac{1}{128}[2r - a^8/(1 + a)^2] \tag{5.63}$$

Putting the values in (5.56) and simplifying, we get

$$a^4(a + 2)^4 = 2r(1 + a)^2$$

or

$$a^2(a + 2)^2 = \sqrt{2r}(1 + a) \tag{5.64}$$

The biquadratic equation (5.64) can be solved by standard procedures. It has two real roots, and the solution given below can be shown to be the only positive one [1].

$$a = -1 + \tfrac{1}{2}\sqrt{b} + \tfrac{1}{2}\sqrt{4 - b + 2\sqrt{2r/b}} \tag{5.65}$$

where

$$b = b_1 + b_2 + \tfrac{4}{3} \tag{5.66}$$

with

$$b_1 = \left(\tfrac{64}{27} + r + \sqrt{r^2 + 128r/27}\right)^{1/3} \tag{5.67}$$

$$b_2 = \left(\tfrac{64}{27} + r - \sqrt{r^2 + 128r/27}\right)^{1/3} \tag{5.68}$$

Using (5.44) in (5.58) to (5.60), P_{11}, P_{12}, and P_{22} may be written as

$$P_{11} = r_0\left(\frac{a}{\tau}\right) \tag{5.69}$$

$$P_{12} = \frac{r_0}{2}\left(\frac{a}{\tau}\right)^2 \tag{5.70}$$

$$P_{23} = \frac{r_0}{8}\left(\frac{a}{\tau}\right)^4 \tag{5.71}$$

From (5.69) to (5.71), it may be noted that for a fixed r_0, P_{11}, P_{12}, and P_{23} are proportional to a power of a/τ. If P_{12} and P_{23} are interpreted as derivatives of autocorrelation functions of position and velocity errors [3, p. 316], then it may be concluded that the value of τ which maximizes the position error variance P_{11} also maximizes the initial slopes of the position and velocity error autocorrelation functions. Hence, Fitzgerald [1] interprets this roughly as a minimization of the memory length of the filter.

5.5.6 Kalman Filter Gains

From (5.10), the Kalman filter gains are given by

$$K_1 = P_{11}/r_0 \tag{5.72}$$

$$K_2 = P_{12}/r_0$$

$$K_3 = P_{13}/r_0$$

5.6 RANDOM WALK ACCELERATION MODEL

A special case of the above solution is found when τ is allowed to approach infinity. In the three-state case, this produces an integrated white noise or "random walk" acceleration (RWA) or a "white jerk" model. In the limit, the solution may be found as:

$$\begin{aligned} P_{11}/r_0 &= 2(q/r_0)^{1/6} \\ P_{12}/r_0 &= 2(q/r_0)^{1/3} \\ P_{13}/r_0 &= (q/r_0)^{1/2} \\ P_{22}/r_0 &= 3(q/r_0)^{1/2} \\ P_{23}/r_0 &= 2(q/r_0)^{2/3} \\ P_{33}/r_0 &= 2(q/r_0)^{5/6} \end{aligned} \tag{5.73}$$

The elements of the gain vector are given by

$$\begin{aligned} K_1 &= 2(q/r_0)^{1/6} \\ K_2 &= 2(q/r_0)^{1/3} \\ K_3 &= (q/r_0)^{1/2} \end{aligned} \tag{5.74}$$

The above results are strictly valid only when the pertinent parameters are time invariant. In other cases, they may be used successfully if the parameters vary slowly enough [1]. As an example, the gains given by (5.74) converted to discrete gains [4, sec. 4.3] have been used with considerable success in missile intercept problems [5]. In such problems, the target maneuvers are approximated by a step change in acceleration: The RWA filter follows such a maneuver with zero steady state error. Fitzgerald has also shown that the RWA model is a theoretically correct one, when the target maneuver is an acceleration step occurring at a random time [6].

Faruqi and Davis [7] present a pseudo steady state solution for the three-state RWA problem, for the case in which r_0 varies with target range in such a way as the radar thermal noise proportional to the sixth power of range when expressed in target displacement unit. For constant range rate, the covariance matrix elements vary with range as given in (5.73) but their actual magnitudes depend on the range rate [1].

5.7 SUMMARY

In this chapter, the two-state ECV model is discussed in Section 5.2 and the steady state solution of Fitzgerald is presented. The solution to the random walk velocity model is obtained as a special case of the ECV model and is given in Section 5.3. The general solution to the ECV filter obtained by Nash is given in Section 5.4. The solutions obtained by Fitzgerald [1] and Nash [2] are identical. The continuous-time ECA target tracking filter is discussed in Section 5.5 and the closed-form solution obtained by Fitzgerald is presented. The solution to the random walk acceleration model is obtained as a special case of the ECA model and is given in Section 5.6.

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6.1 INTRODUCTION

The continuous-discrete-time one-dimensional exponentially correlated velocity and acceleration (ECV and ECA) target tracking filters are discussed in this chapter for discrete position measurements obtained by a track-while-scan radar sensor. Exact closed-form solutions of the steady state ECV and ECA filters are presented.

Gupta and Ahn [1] have obtained the exact closed-form solutions for the discrete ECV and ECA tracking filters without any assumptions based on the system's parameters. Simple process noise matrices with only one nonzero element are considered in Ref. 1 as given by (5.6) for the ECV model and as given by (2.55) for the ECA model, and the steady state characteristics of the filters have been analytically obtained. In the ECV/ECA models considered in Ref. 1, it is assumed that the target velocity/acceleration decays exponentially between measurements with no continuous process noise and undergoes an instantaneous random change at each sampling time. Gupta and Ahn applied Kalman's recursive algorithm and also Vaughan's nonrecursive algorithm [2] to obtain separate solutions for the discrete ECV tracking model. For the ECA model, they demonstrated that Kalman's recursive algorithm fails to yield a closed-form solution. However, Vaughan's nonrecursive algorithm has been applied successfully to obtain solutions for both ECV and ECA models.

Gupta [3] considered a more general process noise matrix with all the elements present for the ECV and ECA models and obtained exact steady state solutions applying Vaughan's nonrecursive algorithm.

In Ref. 4, a closed-form solution for ECV filter is obtained for the most general process noise matrix with known system's parameters.

6.2 ECV TARGET TRACKING FILTER

Consider a vehicle such as an aircraft moving with a random exponentially correlated velocity (ECV) perturbed by a white noise process.

6.2.1 Dynamic Model

If the continuous-time dynamic model of the vehicle described by Eq. (5.1) is sampled at discrete times, then the discrete-time dynamic model may be described by a vector matrix equation of the form [4]

$$X_{n+1} = FX_n + U_n \tag{6.1}$$

where

$$X_n = \begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix} \tag{6.2}$$

$$F = \begin{bmatrix} 1 & \tau(1-e) \\ 0 & e \end{bmatrix} \tag{6.3}$$

The covariance matrix of U_n is assumed to be given by [3, 4]

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \tag{6.4}$$

where

$$q_{11} = \tau^2\sigma_v^2[4e - 3 - e^2 + 2\theta] \quad (6.5)$$
$$q_{12} = \tau\sigma_v^2[1 - e]^2$$
$$q_{22} = \sigma_v^2[1 - e^2]$$

with

$$e = \exp(-\theta) \quad (6.6)$$
$$\theta = T/\tau$$

T is the sampling time, σ_v^2 is the variance of the target velocity, and τ is its correlation time.

6.2.2 Measurement Equation

The position of the target is assumed to be measured by a radar at discrete intervals of time T seconds and all measurements are noisy. The measurement equation is given by

$$x_m(n) = HX_n + v_n \quad (6.7)$$

where

$$H = [1 \quad 0] \quad (6.8)$$

and the variance of v_n is $R = \sigma_x^2$.

6.2.3 Steady State Predicted Covariance Matrix

The steady state predicted covariance matrix $\tilde{P}$ is given by

$$\tilde{P} - Q = F(1 + \tilde{P}H^TR^{-1}H)^{-1}\tilde{P}F^T \quad (6.9)$$

If $\tilde{P}$ is defined as in (2.21), then the normalized covariances may be denoted as

$$\tilde{Y}_{11} = \tilde{P}_{11}/\sigma_x^2 \quad (6.10)$$
$$\tilde{Y}_{12} = \tau(1 - e)\tilde{P}_{12}/\sigma_x^2$$
$$\tilde{Y}_{22} = \tau^2(1 - e)^2\tilde{P}_{22}/\sigma_x^2$$

Then equation (6.9) gives rise to the following three nonlinear equations:

$$\tilde{Y}_{11}^2 - r_1H_1 - 2\tilde{Y}_{12} = H_2 \quad (6.11)$$
$$H_1(\tilde{Y}_{12} - r_2) - e\tilde{Y}_{12} = eH_2 \quad (6.12)$$
$$H_1(\tilde{Y}_{22} - r_3) = e^2H_2 \quad (6.13)$$

where

$$H_1 = 1 + \tilde{Y}_{11} \tag{6.14}$$

$$H_2 = \tilde{Y}_{22}H_1 - \tilde{Y}_{12}^2$$

$$r_1 = f[2\theta - (1 - e)(3 - e)]$$

$$r_2 = f(1 - e)^3$$

$$r_3 = r_2(1 + e)$$

$$f = 1/(r\theta)^2 \tag{6.15}$$

with

$$r = \sigma_x/(\sigma_v T) \tag{6.16}$$

Let

$$\tilde{Y}_{11} = x \tag{6.17}$$

Putting (6.11) in (6.12) and rearranging yields

$$\tilde{Y}_{12} = \frac{ex^2 + (1 + x)r_4}{1 + e + x} \tag{6.18}$$

with

$$r_4 = r_2 - er_1 = f(1 - e^2 - 2e\theta) \tag{6.19}$$

Putting (6.11) in (6.13) and rearranging using (6.18), we get

$$\tilde{Y}_{22} = \frac{e^2}{1 + e + x}\left[\frac{x^2(1 - e + x)}{1 + x} - 2r_4\right] + r_5 \tag{6.20}$$

with

$$r_5 = r_3 - e^2 r_1 = f[(1 - e)^2 + 2e^2(1 - e - \theta)] \tag{6.21}$$

Putting (6.18) and (6.20) in (6.13) and simplifying, we get the following biquadratic equation:

$$x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \tag{6.22}$$

where

$$a_3 = 2a + 2f[a(1 + \theta) - 2\theta] \tag{6.23}$$

$$a_2 = a^2 + r_4^2 + 2f[a(1 + a - \theta) - 2\theta]$$

$$a_1 = 2r_4^2 + 2af[a(1 - \theta) - 2\theta]$$

$$a_0 = r_4^2 - 2af[a(\theta - ae) - 4e(1 + e)]$$

with

$$a = 1 - e^2 \tag{6.24}$$

x may be found by solving (6.22) using standard procedure. Then $\tilde{Y}_{12}$ and $\tilde{Y}_{22}$ are found from (6.18) and (6.20), respectively. It can be shown that the polynomial (6.22) will have only one acceptable real and positive solution [1, 3].

6.2.4 Steady State Filtered Covariance Matrix

Let the normalized steady state elements of the filtered covariance matrix be also defined as in (6.10), replacing tildes by hats on both sides. Then they may be derived in terms of the normalized elements of the $\tilde{P}$ matrix as

$$\begin{aligned} \hat{Y}_{11} &= x/(1+x) \\ \hat{Y}_{12} &= \tilde{Y}_{12}/(1+x) \\ \hat{Y}_{22} &= \tilde{Y}_{22} - \tilde{Y}_{12}^2/(1+x) \end{aligned} \tag{6.25}$$

6.2.5 Steady State Gain Matrix

If the steady state normalized gain elements are written as

$$\begin{aligned} G_1 &= K_1 \\ G_2 &= \tau(1-e)K_2 \end{aligned} \tag{6.26}$$

then they may be shown to be given by

$$\begin{aligned} G_1 &= \hat{Y}_{11} \\ G_2 &= \hat{Y}_{12} \end{aligned} \tag{6.27}$$

Thus the normalized covariances and gains of the ECV filter are expressed only in terms of two independent dimensionless parameters θ and r.

6.3 VAUGHAN'S NONRECURSIVE ALGORITHM

Vaughan [2] derived a nonrecursive algebraic solution for the discrete Riccati equation in which $\tilde{P}_n$ is computed directly from the initial covariance matrix $\tilde{P}_0$. In the steady state ($n \to \infty$), Vaughan established that $\tilde{P}$ is independent of $\tilde{P}_0$. The method of determining the steady state $\tilde{P}$ matrix is as follows [2, 1]:

1. Given the matrices F, H, R, and Q of the model, the Hamiltonian of the system ($n \times n$) is given by

$$K_f = \begin{bmatrix} F^{-T} & F^{-T}H^TR^{-1}H \\ QF^{-T} & F + QF^{-T}H^TR^{-1}H \end{bmatrix} \tag{6.28}$$

2. The eigenvalues of K_f outside the unit circle are determined. If λ is an eigenvalue of K_f, then $1/\lambda$ is also an eigenvalue.
3. Determine the eigenvector matrix W partitioned as

$$W = \begin{bmatrix} W_{11} & \vdots & W_{12} \\ -- & -\vdots- & -- \\ W_{21} & \vdots & W_{22} \end{bmatrix} \tag{6.29}$$

and satisfying

$$WD = K_f W \tag{6.30}$$

where

$$D = \begin{bmatrix} A & \vdots & 0 \\ -- & -\vdots- & -- \\ 0 & \vdots & A^{-1} \end{bmatrix} \tag{6.31}$$

$$A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix} \tag{6.32}$$

4. The steady state $\tilde{P}$ matrix is then given by

$$\tilde{P} = W_{21}W_{11}^{-1} \tag{6.33}$$

6.4 STEADY STATE ECV FILTER BY VAUGHAN'S METHOD

Analytical solution of the ECV target tracking filter based on Vaughan's nonrecursive algorithm is presented in this section.

6.4.1 Characteristic Equation

By Vaughan's algorithm [2], the solution of the filter equation (6.9) is determined by the eigenvectors of the matrix given by (6.28), where F and H are given by (6.3) and (6.8) and Q is given by (6.4). By substituting all the matrices in (6.28), we get

$$K_f = \begin{bmatrix} 1 & 0 & 1/\sigma_x^2 & 0 \\ \tau(1-y) & y & \tau(1-y)/\sigma_x^2 & 0 \\ u_1 & yq_{12} & 1+u_1/\sigma_x^2 & \tau(1-e) \\ u_2 & yq_{22} & u_2/\sigma_x^2 & e \end{bmatrix} \tag{6.34}$$

where

$$u_1 = q_{11} + \tau(1-y)q_{12} = \sigma_v^2\tau^2(b+2\theta) \tag{6.35}$$
$$u_2 = q_{12} + \tau(1-y)q_{22} = \sigma_v^2\tau(2-a)$$

with

$$a = e + y \tag{6.36}$$
$$b = e - y$$
$$y = 1/e$$

The characteristic equation is given by

$$|K_f - I\lambda| = 0 \tag{6.37}$$

where I is a 4×4 identity matrix. By direct evaluation of the determinant equation (6.37), the characteristic polynomial may be obtained as

$$\lambda^4 - \alpha\lambda^3 + \beta\lambda^2 - \alpha\lambda + 1 = 0 \tag{6.38}$$

where

$$\alpha = a + 2 + f(b + 2\theta) \tag{6.39}$$
$$\beta = 2[1 + a + f(b + a\theta)]$$

f is given by (6.15).

6.4.2 Eigenvectors Determination

The eigenvectors corresponding to the eigenvalues λ_i can be determined from the matrix equation given by

$$(K_f - \lambda I)V = 0 \tag{6.40}$$

By direct evaluation of (6.40), the eigenvectors may be found as

$$V = \begin{bmatrix} 1 \\ d_i \\ e_i \\ f_i \end{bmatrix} \tag{6.41}$$

where

$$d_i = \frac{\tau(y-1)\lambda_i}{y-\lambda_i} \tag{6.42}$$

$$e_i = (\lambda_i - 1)\sigma_x^2$$

$$f_i = \frac{\sigma_v^2\tau(2-a)\lambda_i(1+\lambda_i)}{(e-\lambda_i)(y-\lambda_i)}$$

$i = 1, 2.$

6.4.3 Steady State Results

By Vaughan's algorithm the steady state $\tilde{P}$ matrix is now given by (6.33), where W_{11} and W_{21} may be written as

$$W_{11} = \begin{bmatrix} 1 & 1 \\ d_1 & d_2 \end{bmatrix} \tag{6.43}$$

$$W_{21} = \begin{bmatrix} e_1 & e_2 \\ f_1 & f_2 \end{bmatrix} \tag{6.44}$$

The elements of W_{11} and W_{21} are obtained by putting $i = 1, 2$ in (6.42). From (6.33), we have

$$\begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{12} & \tilde{P}_{22} \end{bmatrix} = \frac{1}{d_2 - d_1}\begin{bmatrix} e_1 & e_2 \\ f_1 & f_2 \end{bmatrix}\begin{bmatrix} d_2 & -1 \\ -d_1 & 1 \end{bmatrix} \tag{6.45}$$

From (6.45), the elements of $\tilde{P}$ matrix may be written as

$$\tilde{P}_{11} = (e_1 d_2 - e_2 d_1)/(d_2 - d_1) \tag{6.46}$$

$$\tilde{P}_{12} = (e_2 - e_1)/(d_2 - d_1)$$

$$\tilde{P}_{22} = (f_2 - f_1)/(d_2 - d_1)$$

Using (6.42) in (6.46) and simplifying, the normalized elements of the $\tilde{P}$

matrix may be found as

$$\tilde{Y}_{11} = eS_2 - 1 \tag{6.47}$$
$$\tilde{Y}_{12} = 1 - eS_1 + e^2 S_2$$
$$\tilde{Y}_{22} = \frac{f(e-1)^3[1-(1+a)S_2+S_1]}{e - S_1 + yS_2}$$

where

$$S_1 = \lambda_1 + \lambda_2 \tag{6.48}$$
$$S_2 = \lambda_1 \lambda_2$$

and f is given by (6.15). In Ref. 5, Ekstrand expressed the sum and product of eigenvalues λ_1 and λ_2 in terms of the coefficients of the characteristic polynomial as

$$S_1 = \alpha S_2/(1 + S_2) \tag{6.49}$$
$$S_2 = \tfrac{1}{2}[d + \sqrt{d^2 - 4}]$$
$$d = \tfrac{1}{2}[(\beta - 2) + \sqrt{(\beta + 2)^2 - 4\alpha^2}]$$

where α and β are given by (6.39). Using (6.49), $\tilde{Y}_{11}$, $\tilde{Y}_{12}$, and $\tilde{Y}_{22}$ can be determined from (6.47) without evaluating the eigenvalues. From (6.25), $\hat{Y}_{11}$, $\hat{Y}_{12}$, and $\hat{Y}_{22}$ are given by

$$\hat{Y}_{11} = \tilde{Y}_{11}/(1 + \tilde{Y}_{11}) \tag{6.50}$$
$$\hat{Y}_{12} = \tilde{Y}_{12}/(1 + \tilde{Y}_{11})$$
$$\hat{Y}_{22} = \tilde{Y}_{22} - \tilde{Y}_{12}^2/(1 + \tilde{Y}_{11})$$

From (6.27), the steady state normalized gain elements may be found.

6.4.4 Numerical Results

The steady state normalized covariances and gains are evaluated for the following values of the parameters and the results are given below:

Parameters

$\theta = 0.05$

$r = 0.6$

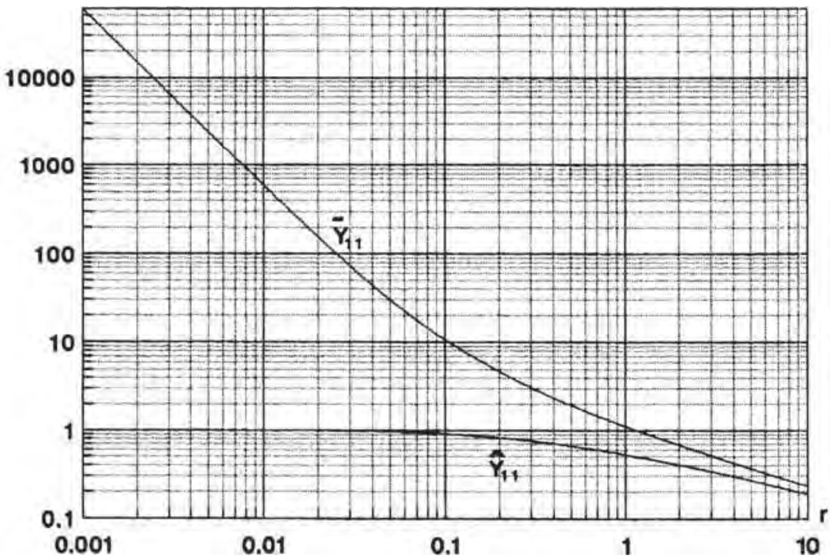

Figure 6.1 Position accuracy before and after measurements.

Computer Results

$$\tilde{Y} = \begin{bmatrix} 1.6587 & 0.7573 \\ 0.7573 & 0.5915 \end{bmatrix}$$

$$\hat{Y} = \begin{bmatrix} 0.6239 & 0.2849 \\ 0.2849 & 0.3758 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.6239 \\ 0.2849 \end{bmatrix}$$

For $\theta = 0.05$, the position accuracy before and after position measurements is plotted against r in Figure 6.1, the velocity accuracy before and after position measurement is plotted against r in Figure 6.2, and the normalized velocity gain against r is shown in Figure 6.3.

6.5 THE DISCRETE ECA TARGET TRACKING FILTER: SINGER'S MODEL

In Singer's model [6], the maneuver equations are derived for the actual continuous-time target motion and are then expressed in discrete time according to the standard discretization procedure, thereby providing accu-

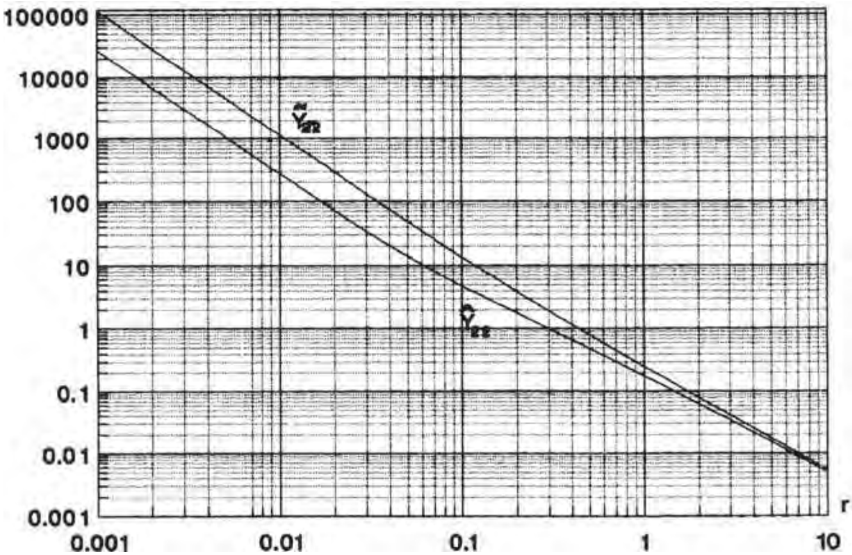

Figure 6.2 Velocity accuracy before and after measurements.

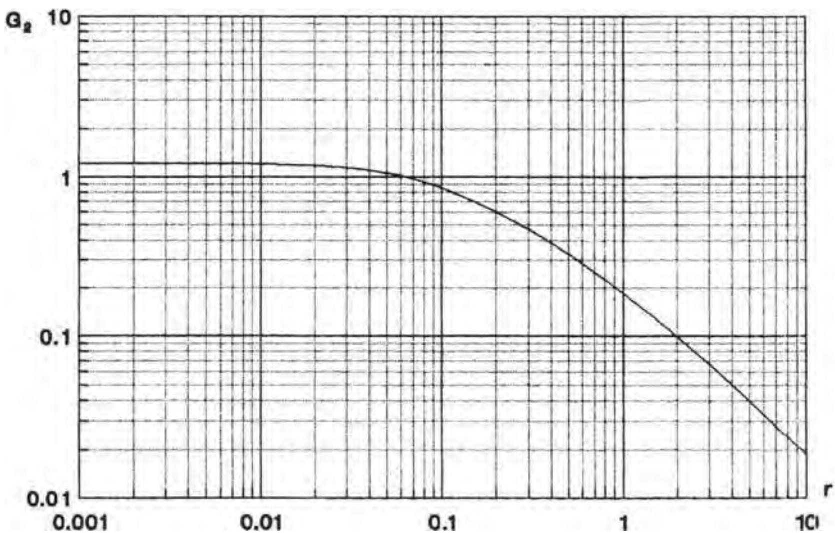

Figure 6.3 Normalized velocity gain as a function of r.

rate statistical representation of the true target behavior. In this way, maneuvering targets are well modeled by Singer assuming a linear acceleration model driven by random noise chosen according to a distribution of potential maneuver accelerations. This filter maintains track through the maneuver and also provides good estimates of position, velocity, and acceleration if the maneuver parameter is correctly chosen.

6.5.1 Dynamic Model

The target equations of motion in one dimension are represented as in (6.1) with

$$X_n = \begin{bmatrix} x_n \\ \dot{x}_n \\ \ddot{x}_n \end{bmatrix} \tag{6.51}$$

and

$$F = \begin{bmatrix} 1 & T & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \tag{6.52}$$

where

$$a_{13} = \tau^2(\theta + e - 1) \tag{6.53}$$

$$a_{23} = \tau(1 - e)$$

$$a_{33} = e$$

$$\theta = T/\tau \tag{6.54}$$

$$e = \exp(-\theta)$$

T is the sampling time and τ is the correlation time of the target acceleration. The process noise covariance Q is expressed as

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \tag{6.55}$$

where

$$q_{11} = \tau^4\sigma_a^2[1 - e^2 + 2\theta(1 - 2e - \theta + \theta^2/3)] \tag{6.56}$$

$$q_{12} = \tau^3\sigma_a^2(1 - e - \theta)^2$$

$$q_{13} = \tau^2\sigma_a^2(1 - e^2 - 2e\theta)$$

$$q_{22} = \tau^2\sigma_a^2[(1 + 2\theta) - (2 - e)^2]$$
$$q_{23} = \tau\sigma_a^2(1 - e)^2$$
$$q_{33} = \sigma_a^2(1 - e^2)$$

σ_a^2 is the variance of the target acceleration.

6.5.2 Measurement Equation

The target position is assumed to be measured at uniform sampling intervals of time T seconds and all measurements are noisy. The measurement equation is given by (6.7) with

$$H = [1 \quad 0 \quad 0] \tag{6.57}$$

and v_n is the additive white noise with variance $R = \sigma_x^2$.

6.5.3 Filtering Equations

The filtering equations for the ECA model are given by (2.50) to (2.54).

6.5.4 Filter Initialization

In Singer's model, the filter is initialized on the basis of the first two position measurements as

$$\hat{x}_2 = x_m(2) \tag{6.58}$$
$$\hat{\dot{x}}_2 = [x_m(2) - x_m(1)]/T$$
$$\hat{\ddot{x}}_2 = 0$$

The corresponding covariance matrix is initialized as

$$\hat{P}_1 = \begin{bmatrix} \sigma_x^2 & \sigma_x^2/T & 0 \\ \sigma_x^2/T & a_1 & a_2 \\ 0 & a_2 & \sigma_a^2 \end{bmatrix} \tag{6.59}$$

where

$$a_1 = \frac{2\sigma_x^2}{T^2} + \frac{\sigma_a^2\tau^2}{2}\left(2 - \theta^2 - 2e - 2e\theta - \frac{2\theta^3}{3}\right) \tag{6.60}$$
$$a_2 = \frac{\sigma_a^2\tau}{\theta}(\theta + e - 1)$$

When the acquisition of the target occurs before the target starts maneuvering, the above covariance initialization reduces to

$$\hat{P}_1 = \begin{bmatrix} \sigma_x^2 & \sigma_x^2/T & 0 \\ \sigma_x^2/T & 2\sigma_x^2/T & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.61}$$

6.5.5 Maneuver Variance Determination

In Singer's model [6], the target acceleration $a(t)$ is modeled as a zero mean random process with exponential autocorrelation

$$R(\tau) = E\{a(t)a(t+\tau)\} = \sigma_a^2 e^{-\alpha|\tau|} \tag{6.62}$$

where σ_a^2 is the variance of the target acceleration and $1/\alpha$ is the time constant of its autocorrelation.

The variance σ_a^2 is given by

$$\sigma_a^2 = \frac{a_M^2}{3}(1 + 4P_M - P_0) \tag{6.63}$$

where P_M is the probability that the target moves with a maximum acceleration a_M (or $-a_M$) and P_0 is the probability that the target has no acceleration.

6.5.6 Special Cases

When θ is small, F reduces to the newtonian matrix of the form given by (2.46) and the process noise covariance matrix reduces to

$$Q = 2\theta\sigma_a^2 \begin{bmatrix} T^4/20 & T^3/8 & T^2/6 \\ T^3/8 & T^2/3 & T/2 \\ T^2/6 & T/2 & 1 \end{bmatrix} \tag{6.64}$$

For a fixed sampling rate, $\tau \to \infty$, Q would reduce to the form given by (2.55).

6.6 FITZGERALD'S STEADY STATE ANALYSIS

The exponentially correlated acceleration (ECA) model of Singer described above is characterized by the following four independent parameters:

τ = correlation time
T = the sampling time
σ_x = rms measurement error

σ_a = rms acceleration

Fitzgerald [7] has shown that Singer's model can be completely specified by only two independent dimensionless parameters if the filter is appropriately normalized. The state variables of the model are redefined to be the three dimensionless quantities as

$$\begin{aligned} z_1 &= x/\sigma_x \\ z_2 &= \dot{x}T/\sigma_x \\ z_3 &= \ddot{x}T^2/\sigma_x \end{aligned} \tag{6.65}$$

Then in the steady state, the rms estimation errors in the new state variables and the dimensionless optimum gains g, h, and k in the Kalman gain vector

$$K = \begin{bmatrix} g \\ h/T \\ 2k/T^2 \end{bmatrix} \tag{6.66}$$

may be shown to depend on only two parameters r_1 and r_2 given by

$$\begin{aligned} r_1 &= \tau/T \\ r_2 &= \sigma_a T^2/\sigma_x \end{aligned} \tag{6.67}$$

Depending upon the situation when r_1 approaches zero, r_2 may be replaced by a parameter r_3 given by

$$r_3 = r_2^2 r_1 = \sigma_a^2 T^3 \tau/\sigma_x^2 \tag{6.68}$$

and when r_1 approaches infinity (random walk acceleration model), r_2 is replaced by a parameter r_4 given by

$$r_4 = r_2^2/r_1 = \sigma_a^2 T^5/(\tau\sigma_x^2) \tag{6.69}$$

The steady state solutions were generated in Ref. 7 for a wide range of r_1 and r_2 by allowing the filter to run until the steady state was reached and the performance of the filter was evaluated. The data presented in Ref. 7 are useful for a preliminary filter design and performance prediction.

6.7 SINGER'S ECA MODEL BASED ON VAUGHAN'S ALGORITHM

If the Q matrix elements given by (6.56) of Singer's model are substituted in the steady state results of the Genaral ECA model of Gupta [3], the steady state characteristics of the Singer's model can be obtained.

6.7.1 Characteristic Polynomial

By putting all matrices in (6.28), the hamiltonian may be found as

$$K_f = \begin{bmatrix} 1 & 0 & 0 & 1/\sigma_x^2 & 0 & 0 \\ -\tau\theta & 1 & 0 & -\tau\theta/\sigma_x^2 & 0 & 0 \\ -\tau^2 a_2 & \tau(1-y) & y & -\tau^2 a_2/\sigma_x^2 & 0 & 0 \\ U_1 & S_1 & yq_{13} & 1 + U_1/\sigma_x^2 & \tau\theta & \tau^2 a_1 \\ U_2 & S_2 & yq_{23} & U_2/\sigma_x^2 & 1 & \tau(1-e) \\ U_3 & S_3 & yq_{33} & U_3/\sigma_x^2 & 0 & e \end{bmatrix} \tag{6.70}$$

where

$$U_1 = -\sigma_a^2\tau^4(b + 2\theta + \theta^3/3) \tag{6.71}$$

$$U_2 = \sigma_a^2\tau^3(a - 2 - \theta^2)$$

$$U_3 = -\sigma_a^2\tau^2(b + 2\theta)$$

$$S_1 = \sigma_a^2\tau^3(2 - a + \theta^2)$$

$$S_2 = \sigma_a^2\tau^2(b + 2\theta)$$

$$S_3 = \sigma_a^2\tau(2 - a)$$

$$a = e + y \tag{6.72}$$

$$b = e - y$$

$$a_1 = \theta - 1 + e$$

$$a_2 = \theta + 1 - y$$

From (6.37), the characteristic polynomial may be obtained as:

$$\lambda^6 - \alpha\lambda^5 + \beta\lambda^4 - \gamma\lambda^3 + \beta\lambda^2 - \alpha\lambda + 1 = 0 \tag{6.73}$$

where

$$\alpha = \alpha_1 - r_2\alpha_3 \tag{6.74}$$

$$\beta = \beta_1 - r_2\beta_3$$

$$\gamma = \gamma_1 - r_2\gamma_3$$

with

$$\alpha_1 = 4 + a \tag{6.75}$$
$$\alpha_3 = b + 2\theta + \theta^3/3$$
$$\beta_1 = 7 + 4a$$
$$\beta_3 = 4b + 2\theta(a + 2) + (a - 4)(\theta^3/3)$$
$$\gamma_1 = 8 + 6a$$
$$\gamma_3 = 6b + 4\theta(1 + a) + 2(1 - 2a)(\theta^3/3)$$

$$r_2 = \frac{1}{(r\theta^2)^2} \tag{6.76}$$
$$r = \frac{\sigma_x}{\sigma_a T^2}$$

6.7.2 Eigenvalues Determination

Let

$$t_i = \lambda_i + 1/\lambda_i \qquad i = 1, 2, 3 \tag{6.77}$$

Then λ_i can be expressed in terms of t_i as

$$\lambda_i = \tfrac{1}{2}\,[t_i \pm \sqrt{(t_i^2 - 4)}] \qquad |\lambda_i| > 1 \tag{6.78}$$

As the inverse of an eigenvalue is also an eigenvalue of K_f, the characteristic polynomial must be of the form given by

$$\prod_{i=1}^{3}(\lambda - \lambda_i)(\lambda - 1/\lambda_i) = 0 \tag{6.79}$$

Expanding (6.79) and comparing it with (6.73), we find

$$t_1 + t_2 + t_3 = \alpha \tag{6.80}$$
$$t_1t_2 + t_2t_3 + t_3t_1 = \beta - 3 \tag{6.81}$$
$$t_1t_2t_3 = \gamma - 2\alpha \tag{6.82}$$

From (6.80) and (6.82), we get

$$t_1, t_2 = \tfrac{1}{2}\,[(\alpha - t_3) \pm \sqrt{(\alpha - t_3)^2 - 4(\gamma - 2\alpha)/t_3}] \tag{6.83}$$

and using (6.81), we get a cubic equation in t_3 as

$$t_3^3 - \alpha t_3^2 + (\beta - 3)t_3 - (\gamma - 2\alpha) = 0 \tag{6.84}$$

Equation (6.84) can be solved for t_3 using standard procedure and then t_1

and t_2 are obtained from (6.83). Knowing t_i, λ_i can be obtained from (6.78). If $|\lambda_i| < 1$, then it is replaced by $1/\lambda_i$ to get the eigenvalues lying outside the unit circle.

6.7.3 Eigenvectors Determination

If λ_i is an eigenvalue of K_f, then its corresponding eigenvector may be found from (6.40) by direct calculation as

$$V = \begin{bmatrix} 1 \\ \tau d_i \\ \tau^2 e_i \\ \sigma_x^2 f_i \\ -\sigma_u^2 \tau^3 g_i \\ \sigma_u^2 \tau^2 h_i \end{bmatrix} \tag{6.85}$$

where

$$d_i = \frac{\theta\lambda_i}{1-\lambda_i} \tag{6.86}$$

$$e_i = \lambda_i \frac{a_3 - a_2\lambda_i}{(1-\lambda_i)(y-\lambda_i)}$$

$$f_i = \lambda_i - 1$$

$$g_i = \frac{\lambda_i(1+\lambda_i)[b_3(1+\lambda_i^2) + b_4\lambda_i]}{(1-\lambda_i)^2(y-\lambda_i)(e-\lambda_i)}$$

$$h_i = \frac{\lambda_i[b_1(1+\lambda_i^2) - b_2\lambda_i]}{(\lambda_i - 1)(y-\lambda_i)(e-\lambda_i)}$$

with

$$b_1 = b + 2\theta \tag{6.87}$$

$$b_2 = 2(b + a\theta)$$

$$b_3 = a - 2 - \theta^2$$

$$b_4 = 2(2-a) + a\theta^2$$

$$a_3 = 1 - y + y\theta$$

$$a_2 = \theta + 1 - y$$

6.7.4 Steady State $\tilde{P}$ Matrix

The steady state $\tilde{P}$ matrix is given by (6.33), where W_{11} and W_{21} are determined by the eigenvectors as

$$W_{11} = \begin{bmatrix} 1 & 1 & 1 \\ -\tau d_1 & -\tau d_2 & -\tau d_3 \\ \tau^2 e_1 & \tau^2 e_2 & \tau^2 e_3 \end{bmatrix} \tag{6.89}$$

$$W_{21} = \begin{bmatrix} \sigma_x^2 f_1 & \sigma_x^2 f_2 & \sigma_x^2 f_3 \\ \tau^3 \sigma_a^2 g_1 & \tau^3 \sigma_a^2 g_2 & \tau^3 \sigma_a^2 g_3 \\ -\tau^2 \sigma_a^2 h_1 & -\tau^2 \sigma_a^2 h_2 & -\tau^2 \sigma_a^2 h_3 \end{bmatrix} \tag{6.90}$$

The inverse of W_{11} may be found to be given by

$$W_{11}^{-1} = \frac{-1}{\tau^3 S} \begin{bmatrix} \tau^3(d_3 e_2 - d_2 e_3) & \tau^2(e_2 - e_3) & \tau(d_2 - d_3) \\ \tau^3(d_1 e_3 - d_3 e_1) & \tau^2(e_3 - e_1) & \tau(d_3 - d_1) \\ \tau^3(d_2 e_1 - d_1 e_2) & \tau^2(e_1 - e_2) & \tau(d_1 - d_2) \end{bmatrix} \tag{6.91}$$

where

$$S = \sum (d_1 e_2 - d_2 e_1) \tag{6.92}$$

In (6.92) and subsequent equations, the summation extends over three terms taken in cyclic order as

$$S = (d_1 e_2 - d_2 e_1) + (d_2 e_3 - d_3 e_2) + (d_3 e_1 - d_1 e_3) \tag{6.93}$$

Let the normalized covariances be defined as:

$$\begin{aligned} \tilde{Y}_{11} &= \tilde{P}_{11}/\sigma_x^2 \\ \tilde{Y}_{12} &= \tau \tilde{P}_{12}/\sigma_x^2 \\ \tilde{Y}_{13} &= \tau^2 \tilde{P}_{13}/\sigma_x^2 \\ \tilde{Y}_{22} &= \tau^2 \tilde{P}_{22}/\sigma_x^2 \\ \tilde{Y}_{23} &= \tau^3 \tilde{P}_{23}/\sigma_x^2 \\ \tilde{Y}_{33} &= \tau^4 \tilde{P}_{33}/\sigma_x^2 \end{aligned} \tag{6.94}$$

By direct evaluation of (6.33), the normalized elements of $\tilde{P}$ matrix may be found as

$$\tilde{Y}_{11} = \frac{1}{S}\sum f_1(d_2e_3 - d_3e_2) \tag{6.95}$$
$$\tilde{Y}_{12} = \frac{1}{S}\sum f_1(e_3 - e_2)$$
$$\tilde{Y}_{13} = \frac{1}{S}\sum f_1(d_3 - d_2)$$
$$\tilde{Y}_{22} = \frac{f}{S}\sum g_1(e_3 - e_2)$$
$$\tilde{Y}_{23} = \frac{f}{S}\sum g_1(d_3 - d_2)$$
$$\tilde{Y}_{33} = \frac{f}{S}\sum h_1(d_2 - d_3)$$

where

$$f = 1/(r\theta^2)^2 \tag{6.96}$$

6.7.5 Steady State $\hat{P}$ Matrix

If the normalized $\hat{P}$ matrix elements are defined as

$$\hat{Y}_{11} = \hat{P}_{11}/\sigma_x^2 \tag{6.97}$$
$$\hat{Y}_{12} = \tau\hat{P}_{12}/\sigma_x^2$$
$$\hat{Y}_{13} = \tau^2\hat{P}_{13}/\sigma_x^2$$
$$\hat{Y}_{22} = \tau^2\hat{P}_{22}/\sigma_x^2$$
$$\hat{Y}_{23} = \tau^3\hat{P}_{23}/\sigma_x^2$$
$$\hat{Y}_{33} = \tau^4\hat{P}_{33}/\sigma_x^2$$

then from (2.56) and (2.58), they may be derived as

$$\hat{Y}_{11} = \tilde{Y}_{11}/H_1 \tag{6.98}$$
$$\hat{Y}_{12} = \tilde{Y}_{12}/H_1$$
$$\hat{Y}_{13} = \tilde{Y}_{13}/H_1$$
$$\hat{Y}_{22} = \tilde{Y}_{22} - \tilde{Y}_{12}^2/H_1$$
$$\hat{Y}_{23} = \tilde{Y}_{23} - \tilde{Y}_{12}\tilde{Y}_{13}/H_1$$
$$\hat{Y}_{33} = \tilde{Y}_{33} - \tilde{Y}_{13}^2/H_1$$

where

$$H_1 = 1 + \tilde{Y}_{11} \tag{6.99}$$

6.7.6 Steady State Gain Vector

From (2.56), the normalized gain elements may be found as

$$\begin{aligned} G_1 &= K_1 = \hat{Y}_{11} \\ G_2 &= TK_2 = \hat{Y}_{12} \\ G_3 &= T^2K_3 = \hat{Y}_{13} \end{aligned} \tag{6.100}$$

Thus all normalized covariances and gains are functions of only two independent dimensionless parameters r and θ.

6.7.7 Numerical Results

The normalized covariances and gains given by (6.95), (6.98), and (6.100) are evaluated for the following values of the parameters and the results are presented below.

Parameters

$$r = 0.4$$
$$\theta = 1.137078$$

Computer Results

$$\tilde{Y} = \begin{bmatrix} 251.1866 & 325.9806 & 137.0729 \\ 325.9806 & 464.6805 & 263.5614 \\ 137.0729 & 263.5614 & 365.3249 \end{bmatrix}$$

$$\hat{Y} = \begin{bmatrix} 0.9960 & 1.2926 & 0.5435 \\ 1.2928 & 43.3127 & 86.3787 \\ 0.5435 & 86.3787 & 290.820 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.9960 \\ 1.2926 \\ 0.5435 \end{bmatrix}$$

If the steady state $\tilde{P}$, $\hat{P}$, and K matrices are evaluated from the Kalman filtering matrix equations and then normalized as given by (6.94), (6.97), and (6.100), we get the same results.

6.8 BEUZIT'S STEADY STATE RESULTS

The exact closed-form solution of the steady state ECA filter is presented in this section. These results are derived by Beuzit [8] based on a comparison between the Kalman and Wiener filter theories.

6.8.1 Steady State Kalman Gain

If the Kalman gain vector is defined as in (2.72), then the normalized gain elements of the ECA filter may be written as

$$\begin{aligned} G_1 &= K_1 \\ G_2 &= \tau K_2 \\ G_3 &= \tau^2 K_3 \end{aligned} \tag{6.101}$$

The steady state normalized elements of the Kalman gain vector may be derived as [8]

$$\begin{aligned} G_3 &= \frac{2(C^3 - g_1 C^2 + g_2 C - g_3)}{g(C-1)} \\ G_2 &= \frac{4(1+C)}{g\theta} - G_3 \\ G_1 &= 1 + \frac{\rho}{eg} \end{aligned} \tag{6.102}$$

where

$$\begin{aligned} \rho &= 1 - g_1 + g_2 - g_3 \\ g &= 1 + g_1 + g_2 + g_3 \\ C &= (1+e)/(1-e) \end{aligned} \tag{6.103}$$

with

$$\begin{aligned} e &= \exp(-\theta) \\ y &= 1/e \end{aligned} \tag{6.104}$$

and

$$\theta = T/\tau$$

g_1, g_2, and g_3 are obtained as follows: Let

$$r = \sigma_x/(\sigma_a T^2) \tag{6.105}$$

and

$$b = 4C(C/12 - C/\theta^2 + 2/\theta^3) \tag{6.106}$$

then g_3 is determined by the relation [8]:

$$g_3^2 = b + 8\theta r^2 C^2 \tag{6.107}$$

g_1 and g_2 are obtained by solving the following two simultaneous equations [8]:

$$g_2^2 - 2g_1g_3 = b_1 \tag{6.108}$$

$$g_1^2 - 2g_2 = 1 + b_2 \tag{6.109}$$

with

$$g_1g_2 > g_3 \tag{6.110}$$

where

$$b_1 = b + b_2 + 8\theta r^2 \tag{6.111}$$

$$b_2 = \tfrac{1}{3} + C^2 - 4/\theta^2 \tag{6.112}$$

Eliminating g_1 between (6.108) and (6.109), we get the following biquadratic in g_2:

$$g_2^4 - 2b_1g_2^2 - 8g_3^2g_2 + [b_1^2 - 4(1 + b_2)g_3^2] = 0 \tag{6.113}$$

g_2 is obtained by solving the biquadratic (6.113), and then g_1 is determined using (6.108) or (6.109). g_1, g_2, and g_3 are real, positive and satisfy the inequality (6.110).

6.8.2 Steady State $\hat{P}$ Matrix

Let the normalized elements of the $\hat{P}$ matrix be defined as in (6.97). Then these normalized covariances are derived in Ref. 8 as

$$\begin{aligned}
\hat{y}_{11} &= G_1 \\
\hat{y}_{12} &= G_2 \\
\hat{y}_{13} &= G_3 \\
\hat{y}_{22} &= b_6 - f[1 + \theta + (1 - y)/\theta] \\
\hat{y}_{23} &= f + d_1[G_2 + eG_3/(1 + e)] \\
\hat{y}_{33} &= f + b_3
\end{aligned} \tag{6.114}$$

where

$$f = \frac{1}{(\theta^2 r)^2} \tag{6.115}$$

$$d_1 = \frac{egG_3}{\rho(1-e)}$$

$$d = \frac{4(1+C)}{g}$$

$$b_3 = \frac{d_1 G_3}{1+e}$$

$$b_4 = G_3[1 + g/\rho - d_1/C] - dd_1$$

$$b_5 = d[eg(G_3 - 1)/\rho - 1]$$

$$b_6 = b_3 + b_4/\theta + b_5/\theta^2$$

6.8.3 Steady State $\tilde{P}$ Matrix

From (2.19), the predicted covariance matrix may be written as

$$\tilde{P} = (I - KH)^{-1}\hat{P} \tag{6.116}$$

If the normalized elements of the $\tilde{P}$ matrix are also defined as in (6.94), then they may be derived as

$$\tilde{Y}_{11} = \hat{Y}_{11}/(1 - G_1) \tag{6.117}$$

$$\tilde{Y}_{12} = \hat{Y}_{12}/(1 - G_1)$$

$$\tilde{Y}_{13} = \hat{Y}_{13}/(1 - G_1)$$

$$\tilde{Y}_{22} = \hat{Y}_{22} + G_2^2/(1 - G_1)$$

$$\tilde{Y}_{23} = \hat{Y}_{23} + G_2 G_3/(1 - G_1)$$

$$\tilde{Y}_{33} = \hat{Y}_{33} + G_3^2/(1 - G_1)$$

Thus the normalized gains and covariances are all expressed in terms of the dimensionless parameters r and θ. It may be noted that θ and r are, respectively, the reciprocals of the quantities r_1 and r_2 defined in (6.67).

The steady state filter characteristics are evaluated for the following values of the parameters:

Parameters

$$r = 2.0$$

$$\theta = 0.1$$

For these values of the dimensionless parameters, g_1, g_2, and g_3 are obtained as

$$g_1 = 6.8088$$

$$g_2 = 22.1798$$

$$g_3 = 35.8085$$

and hence,

$$g = 65.7972$$

$$\rho = -19.4375$$

$$C = 20.0167$$

The normalized gains and covariances may be obtained from (6.102), (6.114), and (6.117) as

$$G = \begin{bmatrix} 0.6735 \\ 3.6655 \\ 9.1111 \end{bmatrix}$$

$$\hat{Y} = \begin{bmatrix} 0.6735 & 3.6655 & 9.1111 \\ 3.6655 & 38.9349 & 155.8860 \\ 9.1111 & 155.8860 & 1097.3300 \end{bmatrix}$$

$$\tilde{Y} = \begin{bmatrix} 2.0629 & 11.2272 & 27.9067 \\ 11.2272 & 80.0887 & 258.1787 \\ 27.9067 & 258.1787 & 1351.5910 \end{bmatrix}$$

To evaluate $\tilde{P}$, $\hat{P}$, and K matrices, let

$$\sigma_x = 0.7040 \text{ nm}$$

and

$$T = 4.0 \text{ s}$$

so that

$$\tau = 40.0 \text{ s}$$

$$\sigma_a = 0.0220 \text{ nm/s}^2$$

Using (6.94), (6.97), and (6.100), the $\tilde{P}$, $\hat{P}$, and K matrices may be obtained as

$$\tilde{P} = \begin{bmatrix} 1.0224 & 0.1391 & 0.0086 \\ 0.1391 & 0.0248 & 0.0020 \\ 0.0086 & 0.0020 & 0.0003 \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} 0.3338 & 0.0454 & 0.0028 \\ 0.0454 & 0.0121 & 0.0012 \\ 0.0028 & 0.0012 & 0.0002 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.6735 \\ 0.0916 \\ 0.0057 \end{bmatrix}$$

If these matrices are evaluated using the Kalman filter matrix equations (2.56) to (2.58), we get the same results for the parameters used.

For $\theta = 0.05$, the position accuracy before and after measurements is plotted against r in Figure 6.4, the velocity accuracy in Figure 6.5, the acceleration accuracy in Figure 6.6. In Figure 6.7 the velocity gain is plotted against r, and in Figure 6.8 the acceleration gain is plotted against r.

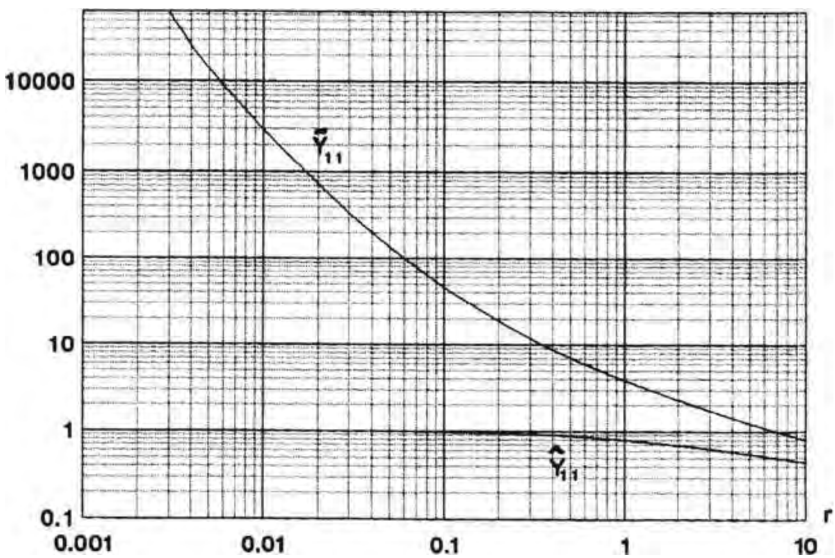

Figure 6.4 Position accuracy before and after measurements.

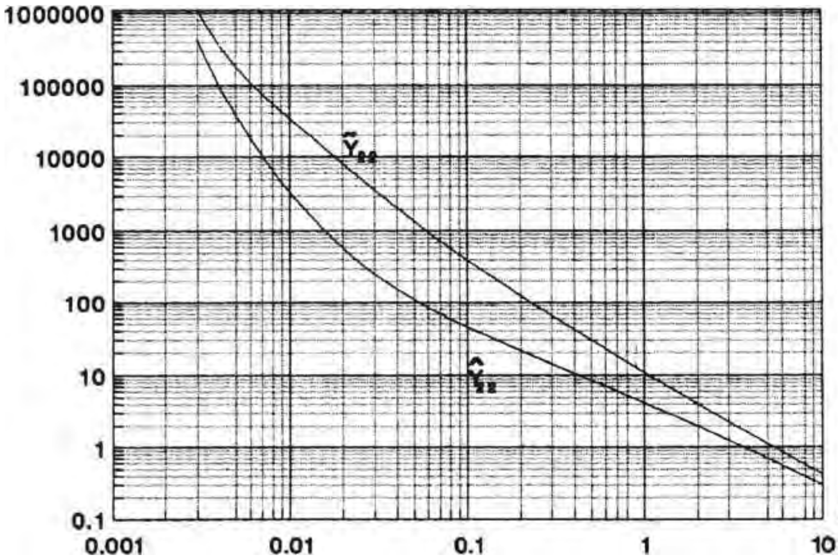

Figure 6.5 Velocity accuracy before and after measurements.

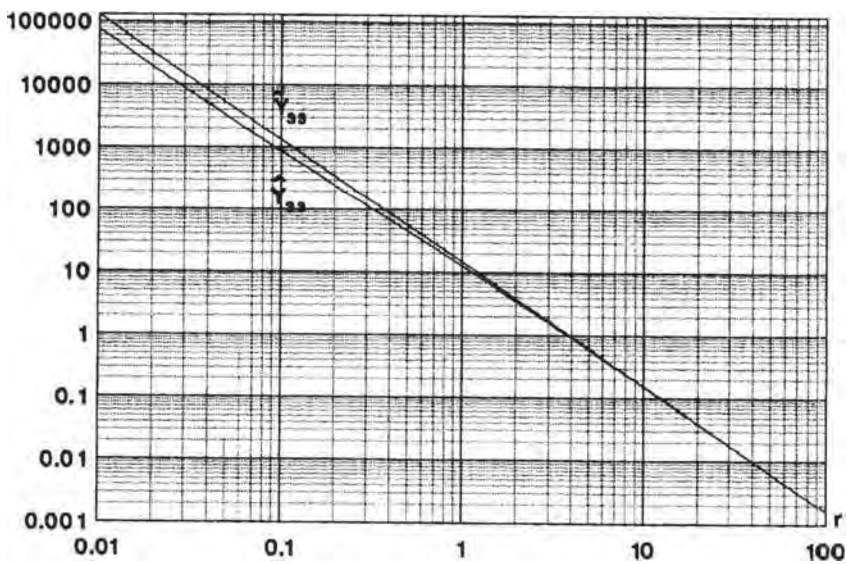

Figure 6.6 Acceleration accuracy before and after measurements.

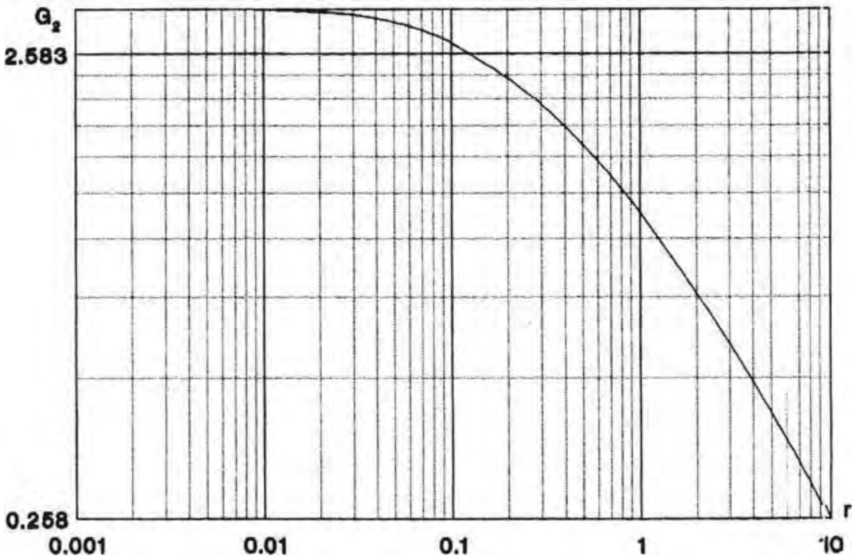

Figure 6.7 Normalized velocity gain as a function of r.

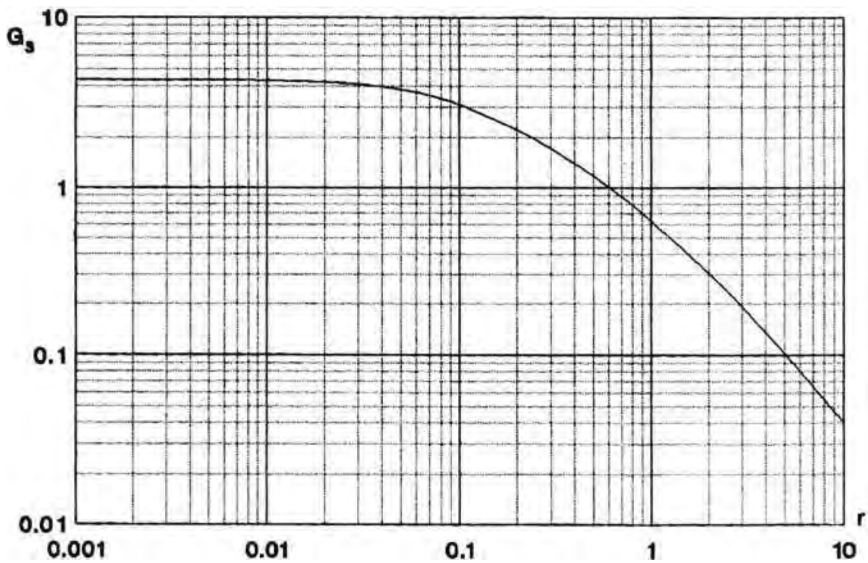

Figure 6.8 Normalized acceleration gain as a function of r.

6.9 SUMMARY

A closed-form solution of the steady state ECV filter obtained by the application of the Kalman recursive algorithm is presented in Section 6.2 for discrete position measurements made by a track-while-scan radar sensor. Vaughan's nonrecursive algorithm for finding the predicted covariance matrix is briefly outlined in Section 6.3. Vaughan's method of obtaining steady state results of the ECV filter is described in Section 6.4. Singer's model of the discrete ECA target tracking filter is presented in Section 6.5. Fitzgerald's steady state analysis of the Singer's model is presented in Section 6.6.

Two methods of analytically determining the steady state characteristics of the Singer's ECA model are presented in Sections 6.7 and 6.8. The first method is based on Vaughan's nonrecursive algorithm, and the second method worked out by Beuzit [8] is based on a comparison between the Kalman and Weiner filter theories. As demonstrated by Fitzgerald [7], the steady state normalized covariances and gains are shown to depend on only two independent dimensionless parameters in both methods. In Ref. 9, the Singer's ECA target tracking filter is extended to two dimensions.

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7

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7.1 INTRODUCTION

Most of the tracking algorithms that have been developed make use of the position measurements only and the use of Doppler measurements has not often been considered in the tracking process. In Refs. 1 to 5 and 7, it has been established in principle that more accurate state estimates are possible by the inclusion of Doppler data.

In a track-while-scan system employing pulsed Doppler such as the moving target detector (MTD) [6], target Doppler is available as part of the measurement process. Experiments with MTD field data have shown that the measured data corresponds unambiguously to the range rate of the target approximately 85 percent of the time. This applies when both a high- and a low-pulse repetition rate (PRF) coherent processing interval are obtained during a single sweep past the target [1]. Simple rules can be formed to reject erroneous Doppler which occurs 15 percent of the time and is due mainly to jet engine and propeller modulations [1].

Thus it is of interest to incorporate valid Doppler data,accurate to a few knots, in the tracking process. Such data adds another dimension to the contact-to-track association process, is an early indication of track maneuvers, and can be used to improve the tracking accuracies [1]. In Ref. 7, some of these advantages are demonstrated via Monte Carlo simulations for a radar system employing two MTD type radars at separated sites.

In the two-state model [1], tracking accuracies for the radial component of motion are computed for the track-while-scan radar system which obtains position and rate data during the dwell time on the target. These results are of practical interest for developing trackers for pulse Doppler surveillance radars.

The normalized accuracies, computed for a two-state Kalman tracking filter with white noise maneuver capability are shown to depend upon two independent dimensionless parameters [1].

In Ref. 1, the general case is described and the filter equations are obtained with position and rate measurements. The corresponding equations for the case when position measurements only are available (the conventional case) are obtained as a special case of the general model. Similarly, the results for the rate measurements only case are obtained as a special case of the general model

By incorporating the rate measurements into the tracking process, Castella [1] has observed that lower steady state tracking errors are obtained and also steady state accuracies are attained much earlier.

In Ref. 1, three simultaneous nonlinear equations are derived for the predicted normalized covariances, and then the filter covariances and gains are computed for different values of parameters numerically via Newton's method. Closed-form steady state solutions for these equations are obtained in Ref. 2 by directly solving the three nonlinear equations, and in Ref. 3, the solution is obtained by making use of Vaughan's nonrecursive algebraic solution for the discrete Ricatti equation [8]. In Ref. 4, it is shown that the two results are identical.

Analytical results for the steady state one-dimensional two-state exponentially correlated velocity target tracking filter [9] is presented in this chapter for discrete position and velocity measurements.

Ramachandra–Mohan–Geetha's model [10], discussed in this chapter, is an extension of Castella's model [1] to the case of a three-state Kalman tracking filter utilizing position and rate measurements. A closed-form steady state solution is obtained for the problem making use of Vaughan's nonrecursive solution for the discrete Ricatti equation [8]. The results for the position measurements only case are obtained as a special case of the general model.

Fitzgerald's steady state analysis of the ECA model [5] with position and rate measurements is presented in this chapter. Fitzgerald has established that the steady state results of the ECA model with position and velocity measurements can be expressed in terms of only three independent dimensionless parameters.

In Ref. 11, the steady state results of Singer's ECA model extended to the case of position and velocity measurements by Fitzgerald [5] are obtained analytically. The results for the position measurements only case are obtained as a special case of the general model.

7.2 CASTELLA'S MODEL: A TWO-STATE TRACKER WITH POSITION AND RATE MEASUREMENTS

Consider a one-dimensional two-state Kalman tracking filter for estimating the range and range rate of a moving target such as an aircraft utilizing both the range and range rate measurements obtained by a track-while-scan radar system which employs pulsed Doppler processing such as a moving target detector providing unambiguous Doppler data. The measurements are obtained at uniform sampling intervals of time T seconds and all measurements are assumed to be noisy.

7.2.1 Dynamic Model

The dynamics of the target is assumed to be described by the vector matrix equation of the form

$$X_{n+1} = FX_n + \omega_n \tag{7.1}$$

The state vector X consists of radial range x_n and range rate $\dot{x}_n$ and is of the form given by (2.4). F is the transition matrix given by (2.5). ω_n is a stationary white noise process with covariance matrix Q_n given by

$$Q_n = E\{\omega_n \omega_n^T\} = qT \begin{bmatrix} T^2/3 & T/2 \\ T/2 & 1 \end{bmatrix} \tag{7.2}$$

q is the spectral density of the continuous white noise process and is equivalent to $\sigma_{\ddot{x}}^2 T$ of Friedland's model given in (2.26). The matrix Q is obtained via the integration of a white noise process [12].

7.2.2 Measurement Model

The measurement model is assumed to be given by

$$Z_n = X_n + V_n \tag{7.3}$$

where

$$Z_n = \begin{bmatrix} x_m(n) \\ \dot{x}_m(n) \end{bmatrix} \tag{7.4}$$

$x_m(n)$ is the measured radial range at scan n and $\dot{x}_m(n)$ is the measured range rate at scan n. As both range and range velocity of the target are measured, the observation matrix H is a 2×2 identity matrix and hence is not shown in (7.3). V_n is the stationary white noise process with covariance matrix R_n given by

$$R_n = E\{v_n v_n^T\} = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_d^2 \end{bmatrix} \tag{7.5}$$

σ_x^2 is the variance of the range measurement error and σ_d^2 is the variance of the range rate (Doppler) measurement error. The range and range rate errors are assumed to be uncorrelated. The maneuver noise ω is assumed to be independent of the measurement noise V.

7.2.3 Filtering Equations

The optimal estimates of the state vector are given by the Kalman filtering algorithm as

$$\hat{X}_n = \tilde{X}_n + K_n(Z_n - \tilde{X}_n) \tag{7.6}$$

$$\tilde{X}_{n+1} = F\hat{X}_n \tag{7.7}$$

The steady state covariances and gain matrices are given by

$$K = \tilde{P}(\tilde{P} + R)^{-1} \tag{7.8}$$

$$\tilde{P} = F\hat{P}F^T + Q \tag{7.9}$$

$$\hat{P} = (I - K)\tilde{P} \tag{7.10}$$

Let the covariance matrices $\tilde{P}$ and $\hat{P}$ be defined as given in (2.21) and (2.33). Let the gain matrix be defined as

$$K_n = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{7.11}$$

Initially, the $\hat{P}_0$ matrix is initialized as

$$\hat{P}_0 = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_d^2 \end{bmatrix} \tag{7.12}$$

On the basis of the first measurement, the initial state vector $\hat{X}_0$ is initialized as

$$\hat{x}_0 = x_m(1) \tag{7.13}$$

$$\hat{\dot{x}}_0 = \dot{x}_m(1)$$

Splitting the covariance equation (7.9) into scalar equations, we get

$$\tilde{P}_{11} = \hat{P}_{11} + 2T\hat{P}_{12} + T^2\hat{P}_{22} + qT^3/3 \tag{7.14}$$

$$\tilde{P}_{12} = \hat{P}_{12} + T\hat{P}_{22} + qT^2/2$$

$$\tilde{P}_{22} = \hat{P}_{22} + qT$$

From (7.7), the states are obtained as

$$\tilde{x}_1 = \hat{x}_0 + \hat{\dot{x}}_0 T \tag{7.15}$$

$$\tilde{\dot{x}}_1 = \hat{\dot{x}}_0$$

From (7.8), the elements of the gain matrix are obtained as

$$K_{11} = [\tilde{P}_{11}(\tilde{P}_{22} + \sigma_d^2) - \tilde{P}_{12}^2]/\Delta \tag{7.16}$$
$$K_{12} = \sigma_x^2 \tilde{P}_{12}/\Delta$$
$$K_{21} = \sigma_d^2 \tilde{P}_{12}/\Delta$$
$$K_{22} = [\tilde{P}_{22}(\tilde{P}_{11} + \sigma_x^2) - \tilde{P}_{12}^2]/\Delta$$

where

$$\Delta = (\tilde{P}_{11} + \sigma_x^2)(\tilde{P}_{22} + \sigma_d^2) - \tilde{P}_{12}^2 \tag{7.17}$$

From (7.10), the elements of the $\hat{P}$ matrix are obtained as

$$\hat{P}_{11} = (1 - K_{11})\tilde{P}_{11} - K_{12}\tilde{P}_{12} \tag{7.18}$$
$$\hat{P}_{12} = (1 - K_{11})\tilde{P}_{12} - K_{12}\tilde{P}_{22}$$
$$\hat{P}_{22} = (1 - K_{22})\tilde{P}_{22} - K_{21}\tilde{P}_{12}$$

Finally from (7.6), the optimal estimates of range and range rate are obtained as

$$\hat{x}_1 = \tilde{x}_1 + K_{11}[x_m(1) - \tilde{x}_1] + K_{12}[\dot{x}_m(1) - \tilde{\dot{x}}_1] \tag{7.19}$$
$$\hat{\dot{x}}_1 = \tilde{\dot{x}}_1 + K_{21}[x_m(1) - \tilde{x}_1] + K_{22}[\dot{x}_m(1) - \tilde{\dot{x}}_1]$$

It may be noted from (7.19) that both position and rate measurements update each element of the state vector.

7.2.4 The Case When Only Position Measurements Are Available

The corresponding filtering equations for the case when only position measurements are available (conventional case) are obtained by letting $\sigma_d \to \infty$ in Eqs. (7.12) to (7.19). The results are

$$\hat{P}_0 = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \infty \end{bmatrix} \tag{7.20}$$

$$\hat{x}_0 = x_m(1) \tag{7.21}$$
$$\hat{\dot{x}}_0 = 0$$

The expressions for $\tilde{P}_{11}$, $\tilde{P}_{12}$, and $\tilde{P}_{13}$ will remain the same as in (7.14) and those of states remain the same as (7.15). The gain matrix elements given by (7.16) become

$$K_{11} = \tilde{P}_{11}/(\tilde{P}_{11} + \sigma_x^2) \tag{7.22}$$
$$K_{12} = 0$$
$$K_{21} = \tilde{P}_{12}/(\tilde{P}_{11} + \sigma_x^2)$$
$$K_{22} = 0$$

For position measurements only case, $\hat{P}$ elements given by (7.18) become

$$\hat{P}_{11} = K_{11}\sigma_x^2 \tag{7.23}$$
$$\hat{P}_{12} = K_{21}\sigma_x^2$$
$$\hat{P}_{22} = \tilde{P}_{22} - K_{21}^2(\tilde{P}_{11} + \sigma_x^2)$$

Finally, the optimal estimates of the state vector given by (7.19) become

$$\hat{x}_1 = \tilde{x}_1 + K_{11}[x_m(1) - \tilde{x}_1] \tag{7.24}$$
$$\hat{\dot{x}}_1 = \tilde{\dot{x}}_1 + K_{21}[x_m(1) - \tilde{x}_1]$$

7.2.5 The Case When Only Rate Measurements Are Available

The corresponding filtering equations for the case when only rate measurements are available are obtained by letting $\sigma_x \to \infty$ in Eqs. (7.12) to (7.19). The results are

$$\hat{P}_n = \begin{bmatrix} \infty & 0 \\ 0 & \sigma_d^2 \end{bmatrix} \tag{7.25}$$
$$\hat{x}_0 = x_m(1) \qquad \text{(guess value)}$$
$$\hat{\dot{x}}_0 = \dot{x}_m(1) \tag{7.26}$$

$x_m(1)$ constitutes a guess value for this case. The expressions for $\tilde{P}_{11}$, $\tilde{P}_{12}$, and $\tilde{P}_{22}$ and also for the states will be the same as (7.14) and (7.15). The gain elements become

$$K_{11} = 0 \tag{7.27}$$
$$K_{12} = \tilde{P}_{12}/(\tilde{P}_{22} + \sigma_d^2)$$
$$K_{21} = 0$$
$$K_{22} = \tilde{P}_{22}/(\tilde{P}_{22} + \sigma_d^2)$$

For range rate measurements only case, the elements of the $\hat{P}$ matrix become

$$\begin{aligned}\hat{P}_{11} &= \tilde{P}_{11} - K_{12}^2(\tilde{P}_{22} + \sigma_d^2) \\ \hat{P}_{12} &= K_{12}\sigma_d^2 \\ \hat{P}_{22} &= K_{22}\sigma_d^2\end{aligned} \tag{7.28}$$

Finally, the optimal estimates of the state vector become

$$\begin{aligned}\hat{x}_1 &= \tilde{x}_1 + K_{12}[\dot{x}_m(1) - \tilde{\dot{x}}_1] \\ \hat{\dot{x}}_1 &= \tilde{\dot{x}}_1 + K_{22}[\dot{x}_m(1) - \tilde{\dot{x}}_1]\end{aligned} \tag{7.29}$$

It may be noticed that there is a perfect symmetry in the results for the two cases.

7.2.6 Steady State Analysis

From (7.9) and (7.10), the combined steady state covariance equation for this model becomes

$$\tilde{P} - Q = F(I - K)\tilde{P}F^T \tag{7.30}$$

If the normalized covariances are defined as

$$\begin{aligned}\tilde{Y}_{11} &= \tilde{P}_{11}/\sigma_x^2 \\ \tilde{Y}_{12} &= \tilde{P}_{12}/(\sigma_x^2/T) \\ \tilde{Y}_{22} &= \tilde{P}_{22}/(\sigma_x^2/T^2)\end{aligned} \tag{7.31}$$

then splitting (7.30) into scalar equations, Castella obtained the following three nonlinear equations for the predicted normalized covariances:

$$[(\tilde{Y}_{11} - A/3)]\,\Delta_1 = \tilde{Y}_{11}(\tilde{Y}_{22} + s^2) - \tilde{Y}_{12}^2 + 2\tilde{Y}_{12}s^2 + \Delta_2 s^2 \tag{7.32}$$

$$(\tilde{Y}_{12} - A/2)\,\Delta_1 = s^2(\tilde{Y}_{12} + \Delta_2) \tag{7.33}$$

$$(\tilde{Y}_{22} - A)\,\Delta_1 = s^2\Delta_2 \tag{7.34}$$

where

$$\Delta_1 = (1 + \tilde{Y}_{11})(s^2 + \tilde{Y}_{22}) - \tilde{Y}_{12}^2 \tag{7.35}$$

$$\Delta_2 = \tilde{Y}_{22}(1 + \tilde{Y}_{11}) - \tilde{Y}_{12}^2 \tag{7.36}$$

with

$$A = (4/r)^2 \tag{7.37}$$

$$r = 4\sigma_x/(\sigma_a T^2) \tag{7.38}$$

$$s = \sigma_a T/\sigma_x \tag{7.39}$$

Castella numerically solved Eqs. (7.32) to (7.34) for different values of r and s and obtained predicted normalized covariances.

7.3 RAMACHANDRA'S STEADY STATE RESULTS

A closed-form steady state solution of this model is obtained in Ref. 2 by solving the nonlinear Eqs. (7.32) to (7.34) algebraically. The results for the steady state normalized covariances and gains are given in this section.

7.3.1 Predicted Normalized Covariances

After considerable algebraic manipulations given in Appendix 7A, the steady state predicted normalized covariances are obtained as [2]:

$$\tilde{Y}_{11} = [2ks^4x + Q_1(A/2 - x)]/(s^2 Q_2) \tag{7.40}$$

$$\tilde{Y}_{12} = Q_1/(2ks^2) \tag{7.41}$$

$$\tilde{Y}_{22} = x + A/2 \tag{7.42}$$

where

$$x = \sqrt{\frac{-d_2 + \sqrt{d_2^2 - 4d_1 d_3}}{2d_1}} \tag{7.43}$$

$$d_1 = B^2 \tag{7.44}$$

$$d_2 = 2BC - s^2k^2(s^2 - 4)$$

$$d_3 = C^2 - 4s^2k^3$$

$$A = (4/r)^2$$

$$B = s^4 + As^2/3 - A \tag{7.45}$$

$$C = Ak(1 + s^2/6) \tag{7.46}$$

$$k = A(s^2 + A/4) \tag{7.47}$$

$$Q_1 = Bx^2 + s^2kx + C \tag{7.48}$$

$$Q_2 = Bx^2 - s^2kx + C \tag{7.49}$$

7.3.2 Steady State Gain Matrix

The steady state components of the gain matrix are obtained as [2]:

$$K_{11} = [2ks^4x - Q_2(A/2 + x)]/(s^2Q_1) \tag{7.50}$$
$$K_{12} = TQ_2/(2ks^4)$$
$$K_{21} = Q_2/(2kTs^2)$$
$$K_{22} = (x - A/2)/s^2$$

7.3.3 Steady State $\hat{P}$ Matrix

The steady state filtered normalized covariances may be found as

$$\hat{Y}_{11} = K_{11} \tag{7.51}$$
$$\hat{Y}_{12} = Q_2/(2ks^2)$$
$$\hat{Y}_{22} = x - A/2$$

Results (7.40) to (7.51) are of practical interest in developing trackers for pulse Doppler surveillance radars. These results eliminate the real time execution of the complete filter equations.

7.4 EKSTRAND'S STEADY STATE RESULTS

A closed-form steady state solution of Castella's model [1] is also obtained in Ref. 3 making use of Vaughan's nonrecursive algebraic solution [8] for the discrete Ricatti equation. The steady state results are given in this section and the details of derivation are given sepatately in Appendix 7B.

7.4.1 Predicted Normalized Covariance Matrix

The steady state solution of Ekstrand for the predicted normalized covariances is given by

$$\tilde{Y}_{11} = \frac{1}{r^2}[\sqrt{\alpha + r^2} + \sqrt{\alpha}]^2\delta - 1 \tag{7.52}$$
$$\tilde{Y}_{12} = \frac{4}{r^2}[\sqrt{\alpha + r^2} + \sqrt{\alpha}]\beta$$
$$\tilde{Y}_{22} = \frac{8}{r^2}[\beta\sqrt{\alpha} + 1]$$

where

$$\alpha = \alpha_1 + \alpha_2 \tag{7.53}$$
$$\alpha_1 = 4/3 + 4/s^2[1 + 2/(3r^2)]$$
$$\alpha_2 = 2\sqrt{(r^2 + 1/3)(1 + 4r_1)(1 + 4r_1/3)}$$
$$\beta = \sqrt{\frac{1 + 4r_1}{1 + 4r_1\alpha}} \tag{7.54}$$
$$\delta = \frac{1 + 16r_1\beta^2}{1 + 8r_1(1 + \beta\sqrt{\alpha})} \tag{7.56}$$

with

$$r_1 = 1/(rs)^2 \tag{7.56}$$

r and s are defined in (7.38) and (7.39).

7.4.2 Filtered Normalized Covariances

The steady state solution of Ekstrand for the filtered normalized covariances is given by

$$\hat{Y}_{11} = \tilde{Y}_{11} - \frac{8}{r^2}(\beta\sqrt{\alpha + r^2} - \tfrac{1}{3}) \tag{7.57}$$
$$\hat{Y}_{12} = \frac{4}{r^2}(\sqrt{\alpha + r^2} - \sqrt{\alpha})\beta$$
$$\hat{Y}_{22} = \frac{8}{r^2}(\beta\sqrt{\alpha} - 1)$$

7.4.3 Steady State Gains

The steady state gains are given by

$$K_{11} = \hat{Y}_{11} \tag{7.58}$$
$$K_{12}/T = \hat{Y}_{12}/s^2$$
$$K_{21}T = \hat{Y}_{12}$$
$$K_{22} = \hat{Y}_{22}/s^2$$

In Figures 7.1 to 7.7 the normalized covariances and gains are plotted as functions of r and s which are in turn functions of the four basic parameters σ_x, σ_a, σ_d, and T. From these figures, one can assess how the accuracy depends on various parameters and also what can be gained by including velocity (Doppler) measurements into the tracking process.

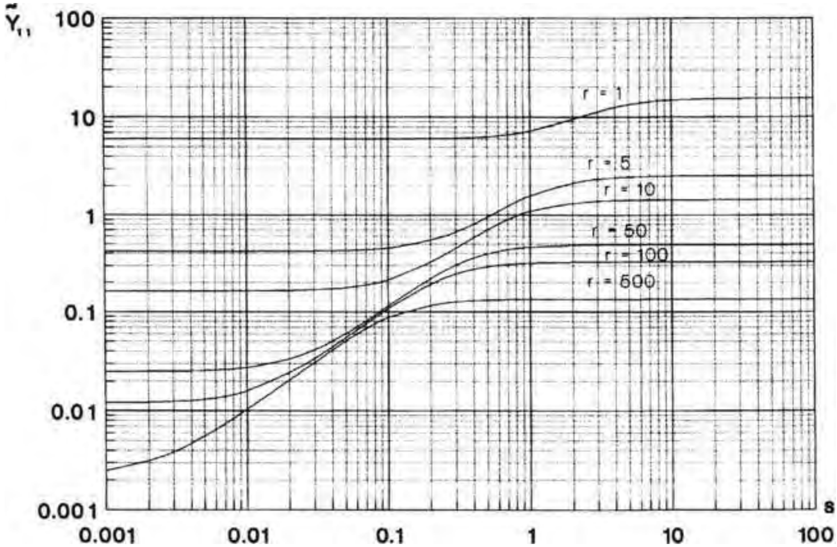

Figure 7.1 Predicted position accuracy as a function of r and s. (From Ref. 3, © 1983 IEEE.)

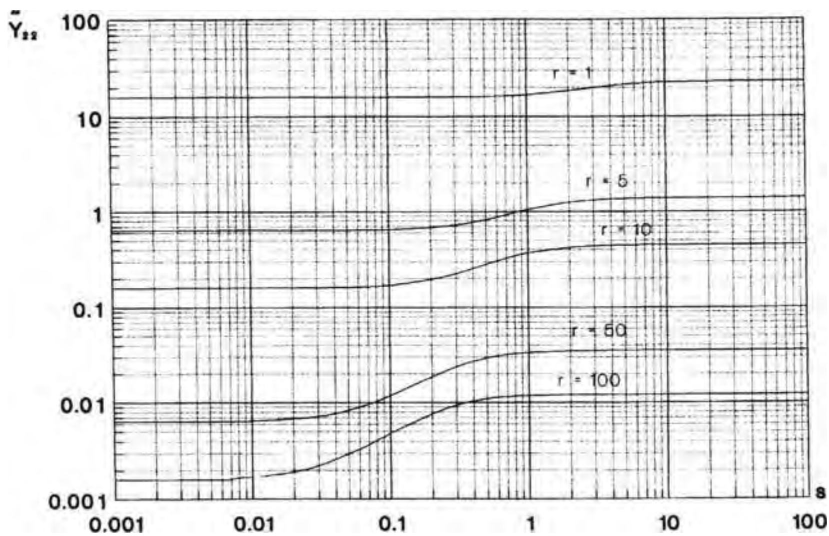

Figure 7.2 Predicted velocity accuracy as a function of r and s. (From Ref. 3, © 1983 IEEE.)

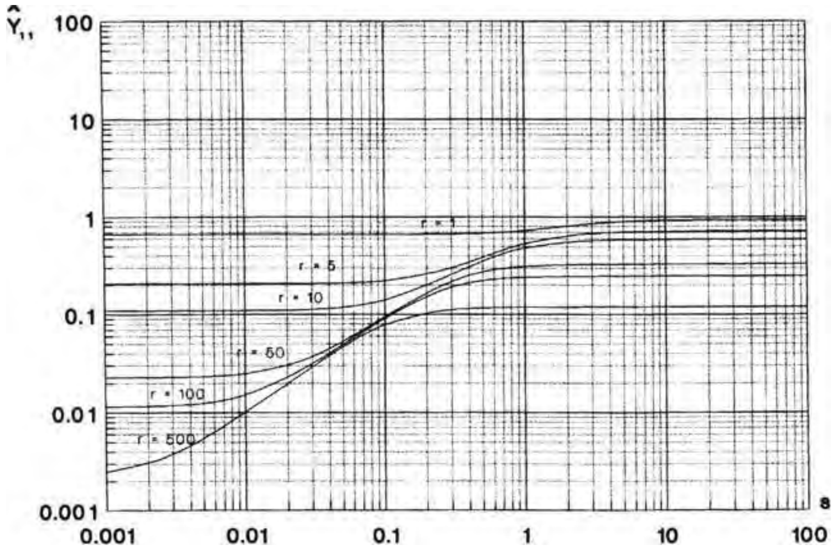

Figure 7.3 Filtered position accuracy as a function of r and s. (From Ref. 3, © 1983 IEEE.)

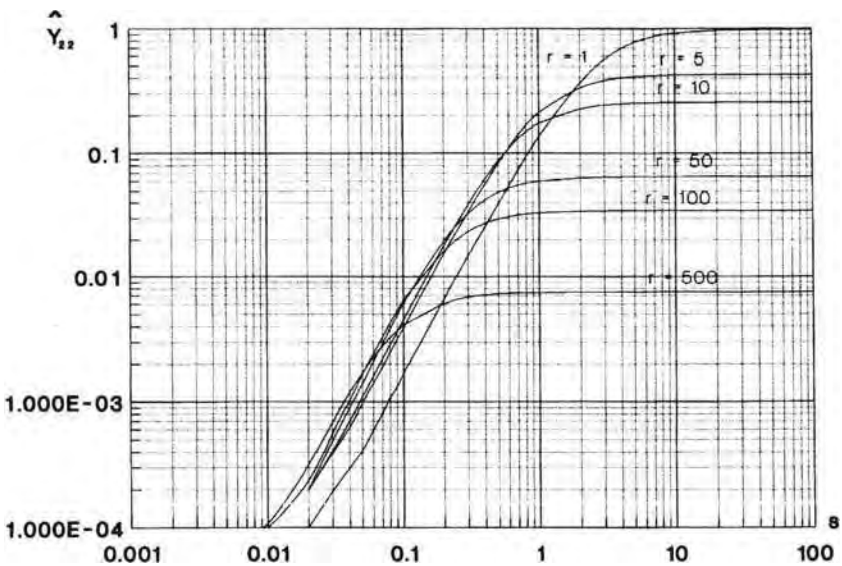

Figure 7.4 Filtered velocity accuracy as a function of r and s. (From Ref. 3, © 1983 IEEE.)

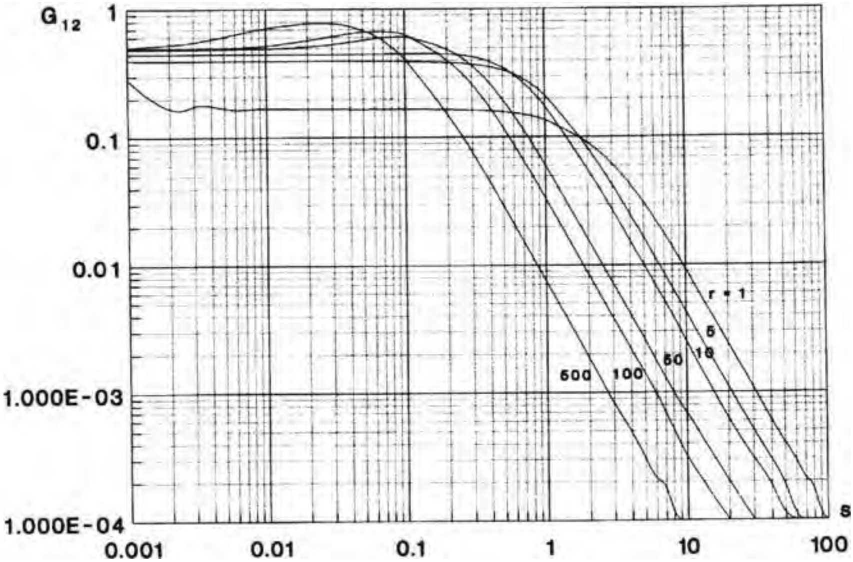

Figure 7.5 G_{12} as a function of r and s. (From Ref. 3, © 1983 IEEE.)

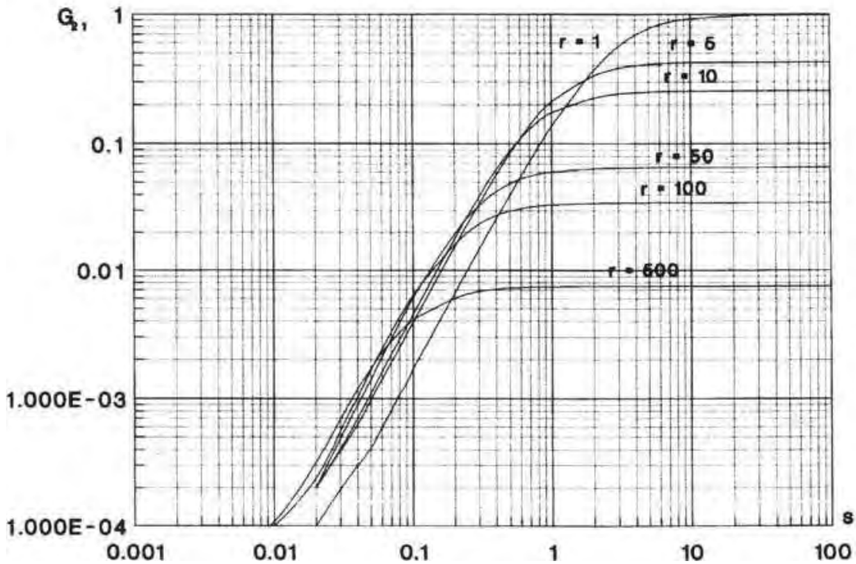

Figure 7.6 G_{21} as a function of r and s. (From Ref. 3, © 1983 IEEE.)

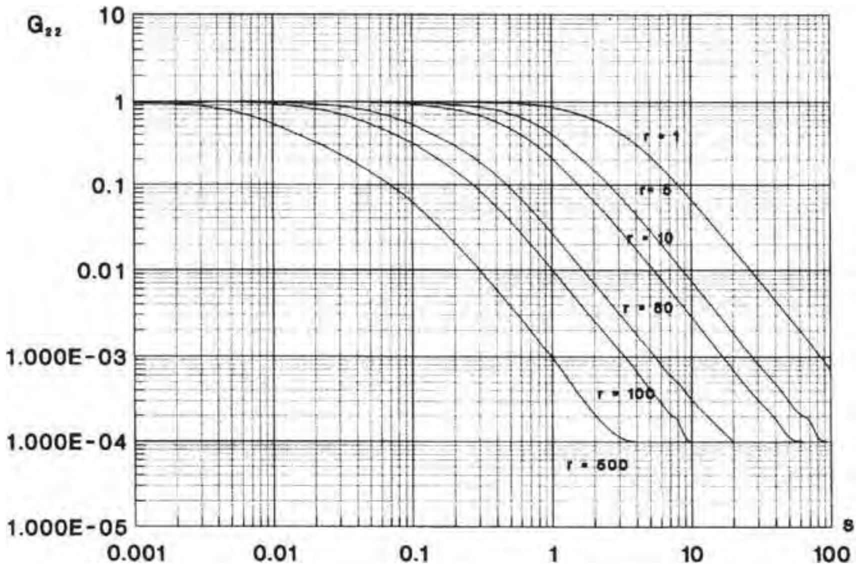

Figure 7.7 G_{22} as a function of r and s. (From Ref. 3, © 1983 IEEE.)

7.4.4 The Case with Position Measurements Only

The case with position measurements only is obtained by letting $s \to \infty$. For this case, we get

$$\alpha = 1 + 1/3 + 2\sqrt{r^2 + 1/3} \tag{7.59}$$
$$\beta = 1$$
$$\gamma = 1$$

Substituting these parameters into (7.52), (7.57), and (7.58) will give the case with range measurements only. It may be seen that by deleting the $\frac{1}{3}$ terms in α, the results coincide with those of Friedland's model discussed in Chapter 2. The main difference between this model and Friedland's model is the one-to-one element of the Q matrix. Obviously, because of this, the terms $\frac{1}{3}$ are involved in the expression for α.

7.5 IDENTICAL STEADY STATE RESULTS

Although the closed-form steady state solutions (7.40) to (7.51) and (7.53) to (7.58) appear to be different for the same model, it is shown in Ref. 4 that they are identical with the following substitutions:

$$\begin{aligned} \alpha &= 16x^2k^2s^4/(AQ_1Q_2) \\ \beta &= \sqrt{Q_1Q_2}/(2\sqrt{A}\,ks^2) \\ \delta &= (s^2 + A/2 - x)/s^2 \\ \alpha + r^2 &= 16(Bx^2 + C)^2/(AQ_1Q_2) \\ \sqrt{\alpha + r^2} + \sqrt{\alpha} &= 4\sqrt{Q_1/(AQ_2)} \\ \beta\sqrt{\alpha} &= 2x/A \end{aligned} \tag{7.60}$$

where the quantities on the left-hand side of (7.60) are as defined in Ekstrand's results [3] and those on the right-hand side are as defined in Ramachandra's results [2].

7.6 ECV TARGET TRACKING FILTER

Consider a one-dimensional two state exponentially correlated velocity target tracking filter making use of discrete position and velocity measurements obtained by a track-while-scan radar system which employs pulsed Doppler processing such as a moving target detector providing unambiguous Doppler data.

7.6.1 Dynamic Model

The dynamic model is the same as that described in (6.1).

7.6.2 Measurement Model

The measurement model is the same as that described in (7.3).

7.6.3 Filtering Equations

The optimal estimates of the state vector are given by the Kalman filtering algorithm as given by (7.6) and (7.7). The steady state covariances and gain matrices are given by (7.8) to (7.10). Let these matrices be defined as in (2.21), (2.33), and (7.11)

7.6.4 Characteristic Equation

By Vaughan's method, the steady state solution of the $\tilde{P}$ matrix is given by (6.33). Putting all the matrices in (6.28), we get

$$K_f = \begin{bmatrix} 1 & 0 & 1/\sigma_x^2 & 0 \\ \tau(1-y) & y & \tau(1-y)/\sigma_x^2 & y/\sigma_d^2 \\ u_1 & yq_{12} & 1+u_1/\sigma_x^2 & \tau(1-e)+yq_{12}/\sigma_d^2 \\ u_2 & yq_{22} & u_2/\sigma_x^2 & e+yq_{22}/\sigma_d^2 \end{bmatrix} \tag{7.61}$$

where u_1, u_2, and y are given by (6.35) and (6.36). q_{12} and q_{22} are given by (6.5). K_f is of order 4 since our system model is of order 2. If λ is an eigenvalue of K_f, then $1/\lambda$ is also an eigenvalue of K_f. Hence the eigenvalue problem is of order 2 only. The characteristic equation is obtained by direct evaluation of (6.37) as the fourth-order polynomial given by

$$\lambda^4 - \alpha\lambda^3 + (2+\beta)\lambda^2 - \alpha\lambda + 1 = 0 \tag{7.62}$$

where

$$\alpha = a + 2 - r_1 b + r_2(b + 2\theta) \tag{7.63}$$

$$\beta = 2[a - r_1 b + r_2(b + a\theta) + r_3 B_1]$$

$$B_1 = 2(2-a) - b\theta \tag{7.61}$$

$$a = e + y \tag{7.62}$$

$$b = e - y$$

$$e = \exp(-\theta) \tag{7.64}$$

$$y = \exp(+\theta)$$

$$\theta = T/\tau$$

$$r_1 = 1/(rs)^2 \tag{7.65}$$

$$r_2 = 1/(r\theta)^2$$

$$r_3 = r_1 r_2$$

$$r = \sigma_x/(\sigma_v T)$$

and s is as defined in (7.39).

7.6.5 Eigenvectors Determination

The eigenvectors corresponding to the eigenvalues λ_i can be determined from (6.40) as

$$V_i = \begin{bmatrix} 1 \\ \tau d_i \\ \sigma_x^2 e_i \\ -\sigma_v^2 \tau(a-2) f_i \end{bmatrix} \tag{7.66}$$

where $i = 1, 2, 3, 4$ and

$$\begin{aligned} d_i &= \frac{\lambda_i(a_1 + a_2\lambda_i)}{\lambda_i^2 - a_3\lambda_i + 1} \\ e_i &= \lambda_i - 1 \\ f_i &= \frac{\lambda_i(1+\lambda_i)}{\lambda_i^2 - a_3\lambda_i + 1} \end{aligned} \tag{7.67}$$

with

$$\begin{aligned} a_1 &= e(y-1)[1 + r_2(y-1)] \\ a_2 &= 1 - y \\ a_3 &= a - br_2 \end{aligned} \tag{7.68}$$

7.6.6 Steady State $\tilde{P}$ Matrix

By Vaughan's algorithm, the steady state $\tilde{P}$ matrix is given by (6.33), where W_{11} and W_{21} are given by the eigenvectors. By (6.33), we have

$$\begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{12} & \tilde{P}_{22} \end{bmatrix} = \frac{1}{\tau(d_2 - d_1)} \begin{bmatrix} \sigma_x^2 e_1 & \sigma_x^2 e_2 \\ -\sigma_v^2\tau(a-2)f_1 & -\sigma_v^2\tau(a-2)f_2 \end{bmatrix} \begin{bmatrix} \tau d_2 & -1 \\ -\tau d_1 & 1 \end{bmatrix} \tag{7.69}$$

Let the normalized elements of the $\tilde{P}$ matrix be defined as in (6.10). Then, from (7.69), they may be derived as:

$$\begin{aligned} \tilde{Y}_{11} &= \frac{S_2}{D}(a_1 S_1 + a_2 S_2 - b_3) - 1 \\ \tilde{Y}_{12} &= \frac{e-1}{D}(S_1^2 + S_2^2 - a_3 S_1 S_2 + b_1 S_2 - a_3 S_1 + 1) \\ \tilde{Y}_{22} &= \frac{y(1-e)^4}{D}[1 + S_1 - (1+a_3)S_2] \end{aligned} \tag{7.70}$$

where

$$D = b_2 S_2 - a_2 S_1 - a_1 \tag{7.71}$$

$$b_1 = a_3^2 - 2 \tag{7.72}$$

$$b_2 = a_1 + a_2 a_3$$

$$b_3 = a_2 + a_1 a_3$$

$$S_1 = \alpha S_2/(1 + S_2) \tag{7.73}$$

$$S_2 = \tfrac{1}{2}(d + \sqrt{d^2 - 4}) \tag{7.74}$$

$$d = \tfrac{1}{2}[\beta + \sqrt{(\beta + 4)^2 - 4\alpha^2}] \tag{7.75}$$

7.6.7 Steady State $\hat{P}$ Matrix Elements

Let the normalized elements of the $\hat{P}$ matrix be defined as in (6.10), replacing tildes by hats. Then they may be derived as:

$$\hat{y}_{11} = \frac{\tilde{y}_{11}(\rho + \tilde{Y}_{22}) - \tilde{Y}_{12}^2}{(1 + \tilde{Y}_{11})(\rho + \tilde{Y}_{22}) - \tilde{Y}_{12}^2} \tag{7.76}$$

$$\hat{y}_{12} = \frac{\rho \tilde{Y}_{12}^2}{(1 + \tilde{Y}_{11})(\rho + \tilde{Y}_{22}) - \tilde{Y}_{12}^2}$$

$$\hat{y}_{22} = \frac{\rho[\tilde{Y}_{22}(1 + Y_{11}) - \tilde{Y}_{12}^2]}{(1 + \tilde{Y}_{11})(\rho + \tilde{Y}_{22}) - \tilde{Y}_{12}^2}$$

where

$$\rho = (1 - e)^2 S^2/\theta^2 \tag{7.77}$$

7.6.8 Steady State Gain Matrix Elements

The elements of the steady state gain matrix may be derived as

$$K_{11} = \hat{P}_{11}/\sigma_x^2 \tag{7.78}$$

$$K_{12} = \hat{P}_{12}/\sigma_d^2$$

$$K_{21} = \hat{P}_{12}/\sigma_x^2$$

$$K_{22} = \hat{P}_{22}/\sigma_d^2$$

7.6.9 The Case with Range Measurements Only

The results for the case with range measurements only is obtained by letting $s \to \infty$. For this case, the characteristic equation (7.62) reduces to (6.38), the eigenvectors (7.66) reduce to (6.42), and $\tilde{Y}$ matrix elements (7.70) reduce to (6.47).

7.7 RAMACHANDRA–MOHAN–GEETHA'S MODEL: A THREE-STATE TRACKER WITH POSITION AND RATE MEASUREMENTS

A one-dimensional three-state Kalman tracker [10] is described in this section for tracking a moving target such as an aircraft. The tracker utilizes both the position and rate measurements obtained by a track-while-scan radar sensor which employs pulsed Doppler processing such as the moving target detector providing unambiguous Doppler data. The steady state filter parameters have been analytically obtained under the assumption of white noise maneuver capability. The numerical computations of these parameters are in excellent agreement with those obtained from the recursive Kalman filter matrix equations. The solution for the case when only the range measurements are available is obtained as a special case of this model. The radar sensor is assumed to measure the range and range rate of the target at uniform sampling intervals of time and both these measurements are corrupted with noise.

7.7.1 Dynamic Equations

The target dynamics is assumed to be described by the vector matrix equation of the form [10]

$$X_{n+1} = FX_n + \omega_n \tag{7.79}$$

F is the transition matrix as defined in (2.46) and X_n is the state vector consisting of the radial range, range rate, and range acceleration components denoted by x_n, $\dot{x}_n$, and $\ddot{x}_n$ respectively. ω_n is a stationary white noise process with covariance matrix Q_n given by

$$Q_n = E\{\omega_n\omega_n^T\} = qT\begin{bmatrix} T^4/20 & T^3/8 & T^2/6 \\ T^3/8 & T^2/3 & T/2 \\ T^2/6 & T/2 & 1 \end{bmatrix} \tag{7.80}$$

q is the spectral density of the continuous white noise change in acceleration process and is equivalent to $\sigma_{\dddot{x}}^2 T$ where $\sigma_{\dddot{x}}^2$ is the variance of the rate of change of acceleration noise as defined in (2.94). The derivation of the Q_n matrix elements of (7.80) is given separately in Appendix 7C.

7.7.2 Measurement Equation

The measurement model is assumed to be described by

$$Z_n = HX_n + V_n \tag{7.81}$$

where

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{7.82}$$

$$Z_n = \begin{bmatrix} x_m(n) \\ \dot{x}_m(n) \end{bmatrix} \tag{7.83}$$

$x_m(n)$ is the measured radial range at scan n and $\dot{x}_m(n)$ is the measured range rate at scan n. V_n is the stationary white noise process with covariance matrix R_n given by

$$R_n = E\{v_n v_n^T\} = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_d^2 \end{bmatrix} \tag{7.84}$$

σ_x^2 is the variance of the range measurement error and σ_d^2 is the variance of the range rate (Doppler) measurement error. The maneuver noise ω is assumed to be independent of the measurement noise V.

7.7.3 Filtering Equations

The optimal estimates of the state vector are given by (3.22) and (2.12). The steady state gain and covariance matrices are given by (2.56) to (2.58).

7.7.4 Steady State $\tilde{P}$ Matrix

Let the steady state covariance matrix $\tilde{P}$ be defined as given in (2.60). Then the normalized elements of $\tilde{P}$ matrix may be defined as

$$\tilde{Y}_{11} = \tilde{P}_{11}/\sigma_x^2 \tag{7.85}$$

$$\tilde{Y}_{12} = \tilde{P}_{12}/(\sigma_x^2/T)$$

$$\tilde{Y}_{13} = \tilde{P}_{13}/(\sigma_x^2/T^2)$$

$$\tilde{Y}_{22} = \tilde{P}_{22}/(\sigma_x^2/T^2)$$
$$\tilde{Y}_{23} = \tilde{P}_{23}/(\sigma_x^2/T^3)$$
$$\tilde{Y}_{33} = \tilde{P}_{33}/(\sigma_x^2/T^4)$$

Making use of the nonrecursive solution for the discrete Riccati equation [8], it can be shown after considerable algebraic calculations (given separately in Appendix 7D) that the normalized elements of the $\tilde{P}$ matrix may be derived as

$$\begin{aligned}
\tilde{Y}_{11} &= \frac{x}{D}(a_1x^2 + a_2x + a_3) \\
\tilde{Y}_{12} &= \frac{2x}{D}[a_1(1 + x^2) + a_4x] \\
\tilde{Y}_{13} &= \frac{4x}{D}(6a_1\alpha + a_5x) \\
\tilde{Y}_{22} &= \frac{2}{D}(2a_1x^3 + a_6x^2 + a_7x - d) \\
\tilde{Y}_{23} &= \frac{8x}{D}(a_8x + a_9) \\
\tilde{Y}_{33} &= \frac{8}{D}(a_{10}x^2 + a_{11}x + a_{12})
\end{aligned} \tag{7.86}$$

where

$$D = d\gamma(1 + x^2) + a_1(1 - 4\gamma)x \tag{7.87}$$

$$\begin{aligned}
a_1 &= 1 + 3\alpha \\
a_2 &= 1.2d\alpha \\
a_3 &= 4\gamma g - a_1(1 + 2.4\alpha) \\
a_4 &= (3\alpha - 1)d \\
a_5 &= (9\alpha + 1)d_1 + (3\alpha - 1)d_2 \\
a_6 &= 3d(4\alpha - 1) \\
a_7 &= 6a_1(1 + 0.2\beta) - 2g \\
a_8 &= (1 + 12\alpha)d_1 + (6\alpha - 1)d_2 \\
a_9 &= 1.5a_1\beta - g \\
a_{10} &= (21\alpha + 1)d_1 + (15\alpha - 1)d_2 \\
a_{11} &= 9a_1(\beta - 4\alpha) - 2g \\
a_{12} &= a_1(d_1 - d_2)
\end{aligned} \tag{7.88}$$

$$d = d_1 + d_2 \tag{7.89}$$

$$d_1 = \sqrt{w + g} \tag{7.90}$$

$$d_2 = \sqrt{w - g}$$

$$w = a_1 + 0.15\beta(14 + 3a_1) \tag{7.91}$$

$$g = 3\sqrt{a_1\beta(1 + 0.2\alpha)(1 + 0.3\beta)} \tag{7.92}$$

$$\alpha = 1/(rs)^2 \tag{7.93}$$

$$\beta = 1/r^2 \tag{7.94}$$

$$\gamma = 1/s^2 \tag{7.95}$$

$$r = 12\sigma_x/(\sigma_a T^3) \tag{7.96}$$

s is as defined in (7.39). x is obtained by the solution of the biquadratic equation given by

$$x^4 - 4d_1x^3 + 6e_1x^2 - 4d_2x + 1 = 0 \tag{7.97}$$

where

$$e_1 = 1 + 0.2\beta - 4\alpha \tag{7.98}$$

7.7.5 Steady State $\hat{P}$ Matrix

Let the normalized elements of the $\hat{P}$ matrix be also defined as given by (7.85) replacing tildes by hats. Then they may be derived as

$$\hat{Y}_{11} = \frac{1}{D}[a_1x^2(x - 4d_1) + b_1x - b_2] \tag{7.99}$$

$$\hat{Y}_{12} = \frac{2}{D}[b_4 - b_3x - a_1x^2(x - 4d_1)]$$

$$\hat{Y}_{13} = -\frac{4}{D}(6a_1\alpha x + b_5)$$

$$\hat{Y}_{22} = \frac{2}{D}(2a_1x^3 - b_6x^2 + b_7x - b_8)$$

$$\hat{Y}_{23} = \frac{8}{D}(b_9x + b_{10})$$

$$\hat{Y}_{33} = \frac{8}{D}(a_{12}x^2 - b_{11}x - b_{12})$$

where

$$b_1 = 4\gamma g + a_1(7 - 21.6\alpha + 1.2\beta) \tag{7.100}$$

$$b_2 = 1.2d\alpha + 4a_1d_2$$

$$b_3 = a_1(5 + 1.2\beta - 24\alpha)$$
$$b_4 = (3\alpha - 1)d + 4a_1 d_2$$
$$b_5 = (3\alpha - 1)d + 2a_1 d_2$$
$$b_6 = 8a_1 d_1 - d$$
$$b_7 = 6a_1(1 + 0.2\beta - 8\alpha) - 2g$$
$$b_8 = 3d_1(4\alpha - 1) + d_2(5 + 36\alpha)$$
$$b_9 = 1.5a_1\beta + g$$
$$b_{10} = (6\alpha - 1)d + 2a_1 d_2$$
$$b_{11} = 9a_1(\beta - 4\alpha) + 2g$$
$$b_{12} = (15\alpha - 1)d + 2a_1 d_2$$

7.7.6 Steady State Gain Matrix

The elements of the steady state Kalman gain matrix are given by

$$K_{11} = \hat{Y}_{11} \tag{7.101}$$
$$K_{12} = \gamma T \hat{Y}_{12}$$
$$K_{21} = \hat{Y}_{12}/T$$
$$K_{22} = \gamma \hat{Y}_{22}$$
$$K_{31} = \hat{Y}_{13}/T^2$$
$$K_{32} = \gamma \hat{Y}_{23}/T$$

From (7.86) to (7.101), it is seen that the normalized covariances and gains are functions of the dimensionless parameters r and s which are functions of the four basic parameters σ_x, σ_d, σ_a, and T. The normalized covariances and gains are plotted in Figures 7.8 to 7.17 as functions of r and s. These plots throw light on how the accuracies depend on different parameters and also the improvement achieved by incorporating velocity measurements into the filter.

The details of derivation of this model are given separately in Appendix 7D.

7.7.7 Results for Range Measurements Only

The results for range measurements only are obtained by letting $s \to \infty$ or $\gamma \to 0$ and $\alpha \to 0$. Hence, for this conventional case, the normalized

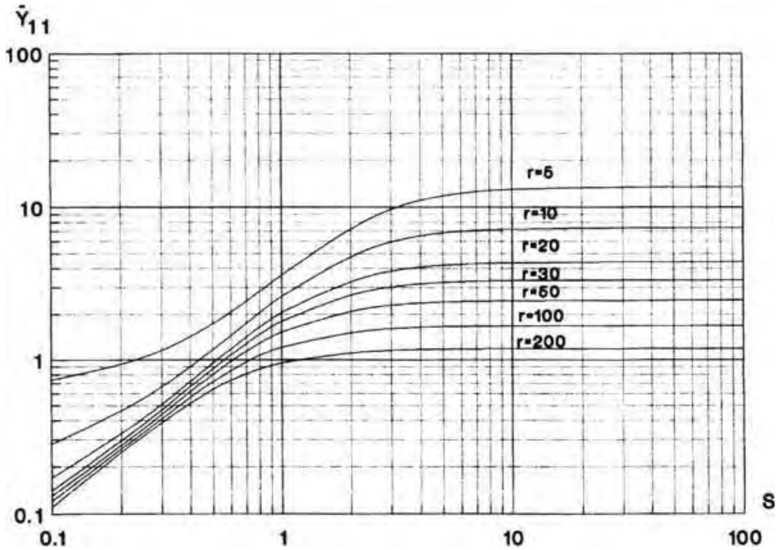

Figure 7.8 Predicted position accuracy as a function of r and S. (From Ref. 10, © 1993 IEEE.)

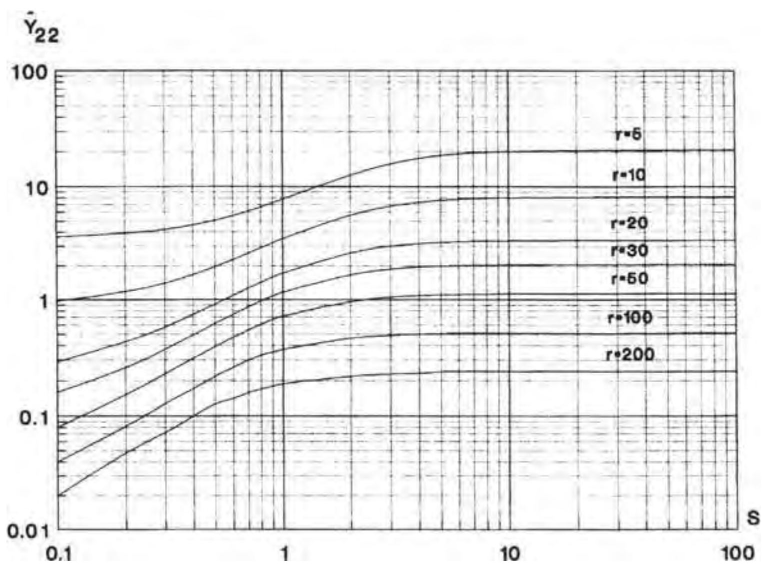

Figure 7.9 Predicted velocity accuracy as a function of r and S. (From Ref. 10, © 1993 IEEE.)

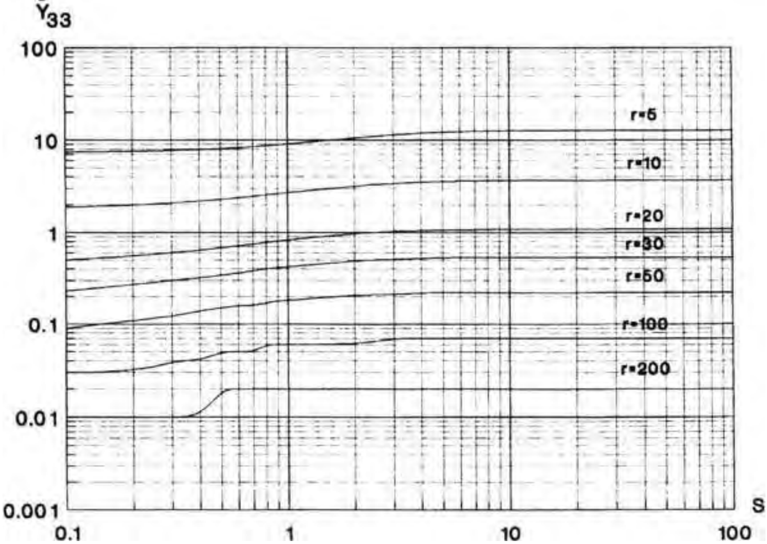

Figure 7.10 Predicted acceleration accuracy as a function of r and S. (From Ref. 10, © 1993 IEEE.)

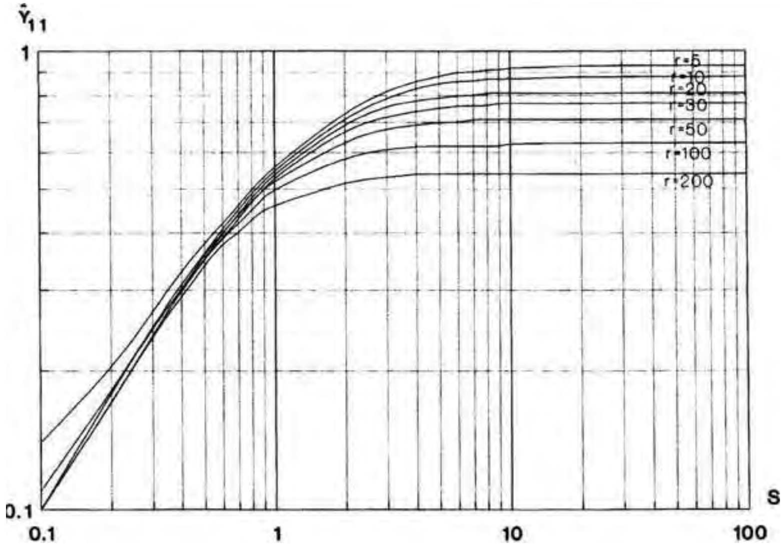

Figure 7.11 Filtered position accuracy as a function of r and S. (From Ref. 10, © 1993 IEEE.)

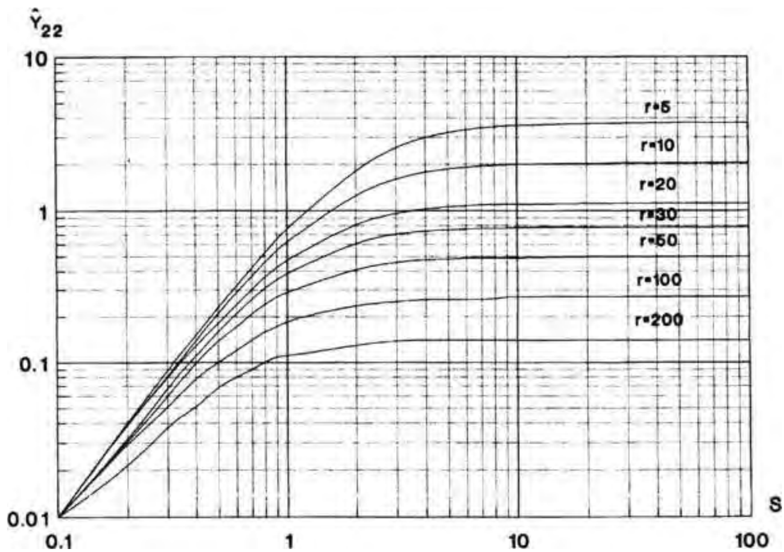

Figure 7.12 Filtered velocity accuracy as a function of r and S. (From Ref. 10, © 1993 IEEE.)

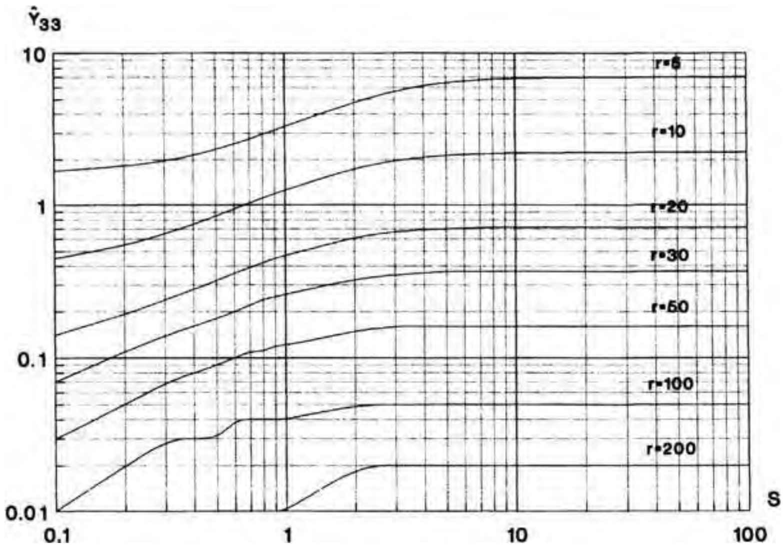

Figure 7.13 Filtered acceleration accuracy as a function of r and S. (From Ref. 10, © 1993 IEEE.)

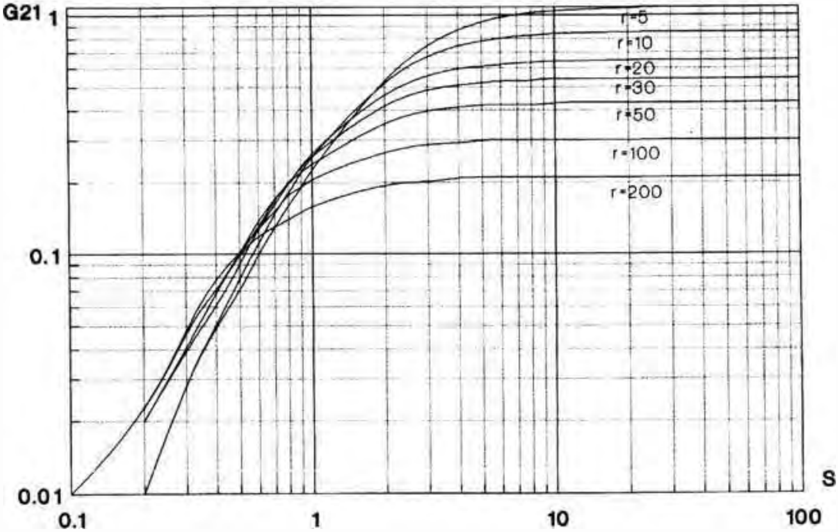

Figure 7.14 G_{21} as a function of r and s. (From Ref. 10, © 1993 IEEE.)

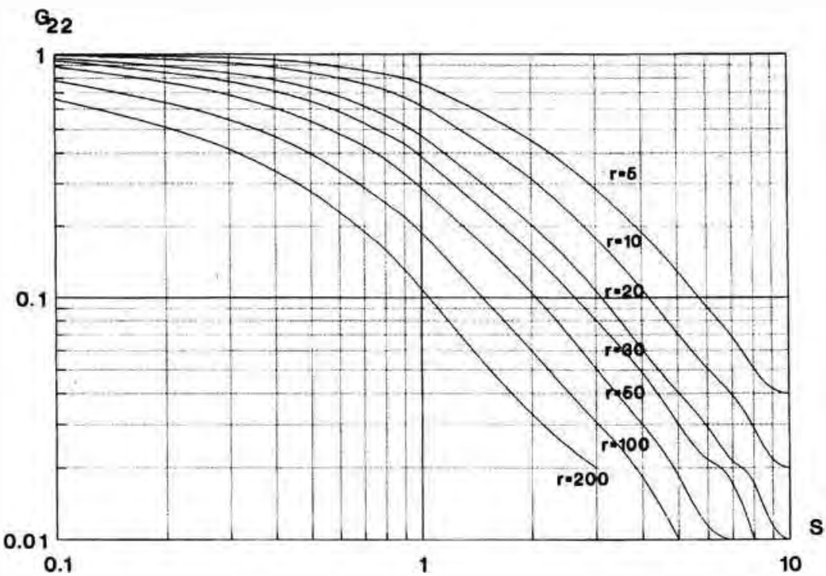

Figure 7.15 G_{22} as a function of r and S. (From Ref. 10, © 1993 IEEE.)

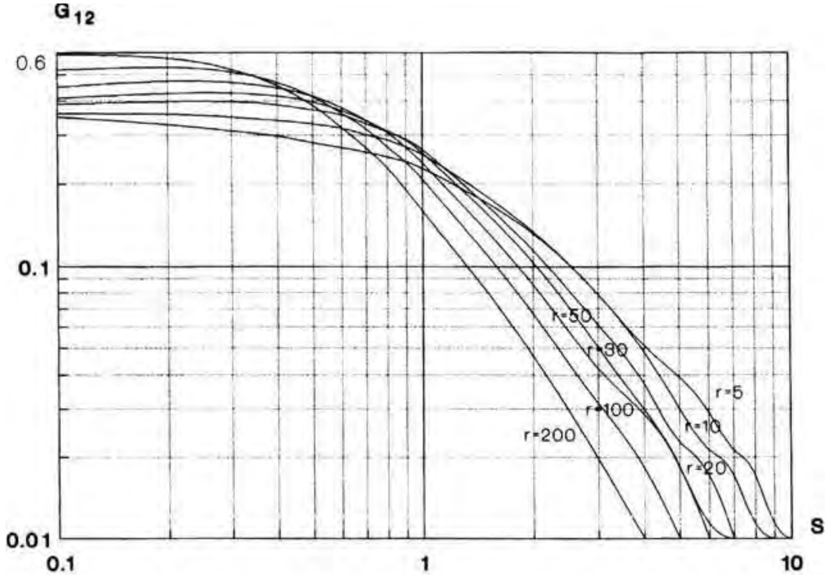

Figure 7.16 G_{12} as a function of r and S. (From Ref. 10, © 1993 IEEE.)

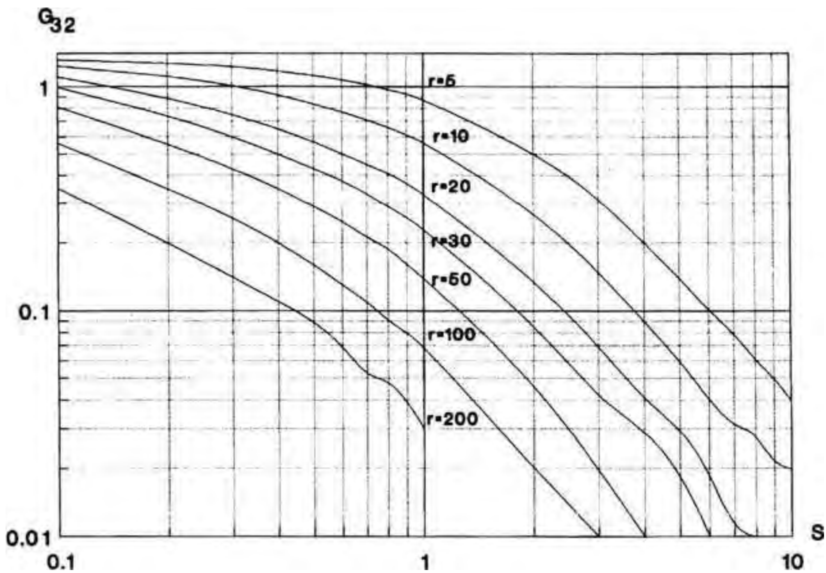

Figure 7.17 G_{32} as a function of r and S. (From Ref. 10, © 1993 IEEE.)

elements of the $\tilde{P}$ matrix as defined in (2.85) are given by

$$\frac{\tilde{P}_{11}}{\sigma_x^2} = x^2 - 1 \tag{7.102}$$

$$\frac{\tilde{P}_{12}}{\sigma_x\sigma_a T^2} = \frac{r}{6}(x^2 - dx + 1)$$

$$\frac{\tilde{P}_{13}}{\sigma_x\sigma_a T} = \frac{rcx}{3}$$

$$\frac{\tilde{P}_{22}}{\sigma_a^2 T^4} = \frac{r^2}{72x}(2x^3 - 3dx^2 + c_1 x - d)$$

$$\frac{\tilde{P}_{23}}{\sigma_a^2 T^3} = \frac{r^2}{18}(cx + c_2)$$

$$\frac{\tilde{P}_{33}}{\sigma_a^2 T^2} = \frac{r^2}{18x}[c(1 + x^2) + c_3 x]$$

where

$$
\begin{aligned}
c &= d_1 - d_2 \\
c_1 &= 6 + 1.2\beta - 2g \\
c_2 &= 1.5\beta - g \\
c_3 &= 9\beta - 2g \\
d_1 &= \sqrt{w + g} \\
d_2 &= \sqrt{w - g} \\
w &= 1 + 2.55\beta \\
g &= 3\sqrt{\beta(1 + 0.3\beta)}
\end{aligned} \tag{7.103}
$$

where β is given by (7.94), and r by (7.96). The normalized elements of the $\hat{P}$ matrix for this special case are also given by (2.85) by replacing tildes by hats on both sides. The normalized elements may be derived as

$$\frac{\hat{P}_{11}}{\sigma_x^2} = \frac{1}{x}(x^3 - 4d_1x^2 + c_4x - 4d_2) \tag{7.104}$$

$$\frac{\hat{P}_{12}}{\sigma_x\sigma_a T^2} = \frac{r}{6x}(3c_{10} - c_5x + 4d_1x^2 - x^3)$$

$$\frac{\hat{P}_{13}}{\sigma_x\sigma_a T} = \frac{rc}{3x}$$

$$\frac{\hat{P}_{22}}{\sigma_a^2 T^4} = \frac{r^2}{72x}(2x^3 - c_6x^2 + c_1x - c_7)$$

$$\frac{\hat{P}_{23}}{\sigma_a^2 T^3} = \frac{r^2}{18x}(c_8 x - c)$$

$$\frac{\hat{P}_{33}}{\sigma_a^2 T^2} = \frac{r^2}{18x}[c(1 + x^2) - c_9 x]$$

where

$$\begin{aligned} c_4 &= 7 + 1.2\beta \\ c_5 &= 5 + 1.2\beta \\ c_6 &= 7d_1 - d_2 \\ c_7 &= 5d_2 - 3d_1 \\ c_8 &= 1.5\beta + g \\ c_9 &= 9\beta + 2g \\ c_{10} &= 3d_2 - d_1 \end{aligned} \tag{7.105}$$

The steady state elements of the gain matrix for this case are given by

$$\begin{aligned} K_{11} &= \hat{Y}_{11} \\ K_{21} &= \hat{Y}_{12}/T \\ K_{31} &= \hat{Y}_{31}/T \end{aligned} \tag{7.106}$$

x is obtained from (7.97) putting $\alpha = 0$ for this case. The numerical computations of $\tilde{P}$, $\hat{P}$, and K matrices from (7.102) to (7.106) are in good agreement with those given by (2.95) to (2.101).

7.8 FITZGERALD'S STEADY STATE ANALYSIS OF ECA MODEL WITH POSITION AND VELOCITY MEASUREMENTS

Fitzgerald's ECA model [5] is a three-state Kalman filter estimating position, velocity, and acceleration of a target. The model assumes that the target behavior may be represented by a random exponentially correlated acceleration. The model utilizes both position and velocity measurements as inputs to the tracking filter.

The filter is of a predictor-corrector type and the filter gains are computed by the Kalman filtering algorithm. The correction operation which simultaneously incorporates the two measurements is of the form given by (3.22), where Z_n is a two-dimensional vector containing the measured position and velocity values as given in (7.81). The filter is characterized by the following five independent parameters:

σ_a = rms target acceleration
τ = correlation time of target acceleration
T = sampling time
σ_x = rms position measurement error
σ_d = rms velocity measurement error

Fitzgerald has shown that the model can be completely specified by only three independent dimensionless parameters if the filter is appropriately normalized. These three parameters are defined as

$$p_1 = \tau/T \tag{7.107}$$

$$p_2 = \sigma_a T^2/\sigma_x$$

$$p_3 = \sigma_x/(\sigma_d T)$$

The filter gains and rms errors are normalized with respect to appropriate powers of T, σ_x, and/or σ_a. For this model, Fitzgerald observed the following:

1. For typical values of p_2, the inclusion of velocity information can yield an improvement of an order of magnitude or more in position estimation
2. There exists a value of p_3 (depending mostly on p_2 and less strongly on p_1), above which the velocity measurements do not help state estimation. There also exists a value of p_3, below which additional velocity measurements accuracy does not provide additional improvements in estimation
3. Although a long time may be required for the filter to reach a complete steady state, it was observed that the velocity errors converge rapidly but the steady state position errors may take much longer time when velocity measurement is used.

7.9 ECA TARGET TRACKING FILTER WITH POSITION AND VELOCITY MEASUREMENTS

In this section, the steady state results of Singer's ECA model extended to the case of position and velocity measurements by Fitzgerald [5] are obtained analytically. The results for the position measurements only case are obtained as a particular case of this general model

7.9.1 Dynamic Model

The vehicle dynamics is of the form given by (6.1), where X_n, F, and Q are as defined by (6.51) to (6.56).

7.9.2 Measurement Model

The measurement equation is the same as that given by (7.81) with H, Z_n, and R given by (7.82) to (7.84).

7.9.3 Filtering Equations

The optimal estimates of the state vector are given by (3.22) and (2.12). The steady state gain and covariance matrices are given by (2.56) to (2.58).

7.9.4 Characteristic Equation

By Vaughan's method, the steady state solution of the $\tilde{P}$ matrix is given by (6.33). By putting all the matrices in (6.28), K_f is obtained as

$$K_f = \begin{bmatrix} 1 & 0 & 0 & 1/\sigma_x^2 & 0 & 0 \\ -\tau\theta & 1 & 0 & -\tau\theta/\sigma_x^2 & 1/\sigma_d^2 & 0 \\ -\tau^2 a_2 & \tau(1-y) & y & -\tau^2 a_2/\sigma_x^2 & \tau(1-y)/\sigma_d^2 & 0 \\ U_1 & S_1 & yq_{13} & 1+U_1/\sigma_x^2 & \tau\theta + yq_{13}/\sigma_d^2 & \tau^2 a_1 \\ U_2 & S_2 & yq_{23} & U_2/\sigma_x^2 & 1+yq_{23}/\sigma_d^2 & \tau(1-e) \\ U_3 & S_3 & yq_{33} & U_3/\sigma_x^2 & yq_{33}/\sigma_d^2 & e \end{bmatrix} \tag{7.108}$$

where U_1, U_2, U_3, S_1, S_2, and S_3 are as given by (6.71).

K_f is of order 6 since our system model is of 3. If λ is an eigenvalue of K_f, then $1/\lambda$ is also an eigenvalue of K_f. Hence the eigenvalue problem is of order 3 only. The characteristic equation is obtained by direct evaluation of (6.37) as the sixth order polynomial of the form given by (6.73), where

$$\alpha = \alpha_1 + r_1\alpha_2 - r_2\alpha_3 \tag{7.109}$$

$$\beta = \beta_1 + r_1\beta_2 - r_2\beta_3 - r_3\beta_4$$

$$\gamma = \gamma_1 + r_1\gamma_2 - r_2\gamma_3 - r_3\gamma_4$$

with

$$\alpha_1 = 4 + a \tag{7.110}$$
$$\beta_1 = 7 + 4a$$
$$\gamma_1 = 8 + 6a$$
$$\alpha_2 = b + 2\theta \tag{7.111}$$
$$\beta_2 = 4b + 2\theta(a + 2)$$
$$\gamma_2 = 6b + 4\theta(a + 1)$$

$$\alpha_3 = \alpha_2 + \frac{\theta^3}{3} \tag{7.112}$$
$$\beta_3 = \beta_2 + \frac{\theta^3}{3}(a - 4)$$
$$\gamma_3 = \gamma_2 + \frac{2\theta^3}{3}(1 - 2a)$$

$$\beta_4 = 4(a - 2 + b\theta) + 2a\theta^2 + \frac{\theta^3}{3}(b - \theta) \tag{7.113}$$
$$\gamma_4 = 8(a - 2 + b\theta + \theta^2) + \frac{\theta^3}{3}(4b + a\theta)$$
$$a = e + y \tag{7.114}$$
$$b = e - y$$
$$e = \exp(-\theta) \tag{7.115}$$
$$y = \exp(+\theta)$$
$$r_1 = \frac{1}{(rs\theta)^2} \tag{7.116}$$
$$r_2 = \frac{1}{(r\theta^2)^2} \tag{7.117}$$
$$r_3 = r_1 r_2 \tag{7.118}$$
$$r = \frac{\sigma_x}{(\sigma_a T^2)} \tag{7.119}$$

s and θ are as defined in (7.39) and (7.64).

7.9.5 Eigenvalues Determination

The eigenvalues are determined by (6.78) to (6.84) as given in Section 6.7.2.

7.9.6 Eigenvectors Determination

If λ_i is an eigenvalue of K_f, then its corresponding eigenvector V_i may be found by direct evaluation of (6.40) as given by

$$V_i = \begin{bmatrix} 1 \\ -\tau d_i \\ \tau^2 e_i \\ \sigma_x^2 f_i \\ \sigma_a^2 \tau^3 g_i \\ -\sigma_a^2 \tau^2 h_i \end{bmatrix} \tag{7.120}$$

where

$$d_i = \frac{\lambda_i}{D_i}(\theta\lambda_i^3 - a_3\lambda_i^2 + a_4\lambda_i - a_5) \tag{7.121}$$

$$e_i = \frac{\lambda_i}{D_i}(a_6\lambda_i^3 + a_7\lambda_i^2 + a_8\lambda_i + a_9)$$

$$f_i = \lambda_i - 1$$

$$g_i = \frac{\lambda_i}{D_i}(a_{10}\lambda_i^3 + a_{11}\lambda_i^2 + a_{11}\lambda_i + a_{10})$$

$$h_i = \frac{\lambda_i}{D_i}(a_{12}\lambda_i^3 - a_{13}\lambda_i^2 + a_{13}\lambda_i - a_{12})$$

with

$$D_i = \lambda_i^4 - a_1\lambda_i^3 + 2a_2\lambda_i^2 - a_1\lambda_i + 1 \tag{7.122}$$

$$a_1 = 2 + a + r_1(b + 2\theta) \tag{7.123}$$

$$a_2 = 1 + a + r_1(b + a\theta)$$

$$a_3 = \theta(1 + a) + r_1(a - 2 + b\theta + \theta^2) \tag{7.124}$$

$$a_4 = \theta a_2$$

$$a_5 = \theta + r_1(2 - a + \theta^2)$$

$$a_6 = y - \theta - 1 \tag{7.125}$$

$$a_7 = (1 + e - 2y) + \theta(1 + a) + r_1[2(a - 2) + \theta(2 + e - 3y) + \theta^2(1 + y)]$$

$$a_8 = (1 + y - 2e) - \theta(1 + a) + r_1[2(2 - a) + \theta(2 + y - 3e) - \theta^2(1 + e)]$$

$$a_9 = \theta + e - 1$$

$$a_{10} = a - 2 - \theta^2 \tag{7.126}$$

$$a_{11} = 2 - a + \theta^2(a - 1)$$

$$a_{12} = b + 2\theta \tag{7.127}$$
$$a_{13} = 3b + 2\theta(1 + a) + r_1[4(a - 2 + b\theta) + \theta^2(a + 2)]$$

7.9.7 Steady State $\tilde{P}$ Matrix

The steady state $\tilde{P}$ matrix is given by (6.33), where W_{11} and W_{21} are determined by the eigenvectors as (7.120). If the normalized elements of $\tilde{P}$ matrix are defined as (6.94), then they are given by (6.95), where the eigenvectors are determined by (7.120) to (7.127).

$$f = r_2 \tag{7.128}$$

7.9.8 Steady State $\hat{P}$ Matrix

If the normalized steady state $\hat{P}$ matrix elements are defined as in (6.97), they are given by

$$\hat{Y}_{11} = \frac{1}{\Delta}[\tilde{Y}_{11}(\tilde{Y}_{22} + s^2) - \tilde{Y}_{12}^2] \tag{7.129}$$
$$\hat{Y}_{12} = \frac{1}{\Delta}(s^2 \tilde{Y}_{12})$$
$$\hat{Y}_{13} = \frac{1}{\Delta}[\tilde{Y}_{13}(\tilde{Y}_{22} + s^2) - \tilde{Y}_{12}\tilde{Y}_{23}]$$
$$\hat{Y}_{22} = \frac{s^2}{\Delta}[\tilde{Y}_{22}(\tilde{Y}_{11} + 1) - \tilde{Y}_{12}^2]$$
$$\hat{Y}_{23} = \frac{s^2}{\Delta}[\tilde{Y}_{23}(\tilde{Y}_{11} + 1) - \tilde{Y}_{12}\tilde{Y}_{13}]$$
$$\hat{Y}_{33} = \tilde{Y}_{33} - \frac{E}{\Delta}$$

where

$$\Delta = (1 + \tilde{Y}_{11})(\tilde{Y}_{22} + s^2) - \tilde{Y}_{12}^2 \tag{7.130}$$
$$E = \tilde{Y}_{13}^2(\tilde{Y}_{22} + s^2) + \tilde{Y}_{23}^2(1 + \tilde{Y}_{11}) - 2\tilde{Y}_{12}\tilde{Y}_{13}\tilde{Y}_{23}$$

7.9.9 Steady State K Matrix

If the normalized elements of the gain matrix are defined as

$$G_{11} = K_{11} \tag{7.131}$$
$$G_{12} = K_{12}/T$$

$$G_{21} = TK_{21}$$
$$G_{22} = K_{22}$$
$$G_{31} = T^2K_{31}$$
$$G_{32} = TK_{32}$$

then

$$\begin{aligned} G_{11} &= \hat{Y}_{11} \\ G_{12} &= \hat{Y}_{12}/s^2 \\ G_{21} &= \hat{Y}_{12} \\ G_{22} &= \hat{Y}_{22}/s^2 \\ G_{31} &= \hat{Y}_{13}/s^2 \\ G_{32} &= \hat{Y}_{23}/s^2 \end{aligned} \tag{7.132}$$

7.9.10 Results for Position Measurements Only Case

The results for the case when only the range measurements are available are obtained as a special case of this general ECA model by letting $s \to \infty$. For this case we get from (7.116) and (7.118),

$$\begin{aligned} r_1 &\to 0 \\ r_3 &\to 0 \end{aligned} \tag{7.133}$$

and the results (7.108) to (7.119) reduce (6.74) to (6.76).

Eigenvalues corresponding to this case are found from (6.78) and the corresponding eigenvectors are determined from (7.120) where

$$\begin{aligned} a_1 &= 2 + a \\ a_2 &= 1 + a \\ a_3 &= \theta(1 + a) \\ a_4 &= \theta a_2 \\ a_5 &= \theta \\ a_6 &= y - \theta - 1 \\ a_7 &= 1 + e - 2y + \theta(1 + a) \\ a_8 &= 1 + y - 2e - \theta(1 + a) \end{aligned} \tag{7.134}$$

$$a_9 = e + \theta - 1$$

$$a_{10} = a - 2 - \theta^2$$

$$a_{11} = 2 - a + \theta^2(a - 1)$$

$$a_{12} = b + 2\theta$$

$$a_{13} = 3b + 2\theta(1 + a)$$

It can be verified that, with these substitutions, the eigenvectors given by (7.121) reduce to those given by (6.86). The normalized $\tilde{P}$ matrix elements can now be obtained from (6.95). The normalized filtered covariances and gains may be found from (6.98) to (6.100).

7.10 SUMMARY

In a track-while-scan system employing pulsed Doppler such as the moving target detector, target Doppler is available as part of the measurement process. A two-state model for estimating position and velocity making use of position and Doppler information is discussed in Section 7.2. The steady state results for this model obtained by directly solving the nonlinear equations are given in Section 7.3, and those obtained by making use of Vaughan's nonrecursive solution are given in Section 7.4. These two results are shown to be identical in Section 7.5. An ECV target tracking model is presented in Section 7.6. A three-state Kalman filter utilizing position and velocity information is discussed in Section 7.7. In Section 7.8, Fitzgerald's steady state analysis of the ECA model with position and velocity measurements is discussed. The steady state results of this model are analytically obtained in Section 7.9.

The derivation of Ramachandra's steady state results of Castella's model is given in Appendix 7A. The derivation of Ekstrand's steady state results of Castella's model is given in Appendix 7B. The derivation of steady state results of Ramachandra-Mohan-Geetha's model is given in Appendix 7D. The derivation of the Q matrix of this model is given in Appendix 7C. The values of symmetric functions are given in Appendix 7E.

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APPENDIX 7A: DERIVATION OF RAMACHANDRA'S STEADY STATE RESULTS

Using (7.36) in (7.35),

$$\Delta_2 = \Delta_1 - s^2(1 + \tilde{Y}_{11}) \tag{A1}$$

Using (7.34) in (7.33) and rearranging terms,

$$\Delta_1 = \frac{s^2 \tilde{Y}_{12}}{\tilde{Y}_{11} - \tilde{Y}_{22} + A/2} \tag{A2}$$

Putting (7.35) and (7.36) in (7.34) and rearranging the terms,

$$1 + \tilde{Y}_{11} = \frac{\tilde{Y}_{12}^2(\tilde{Y}_{22} - A - s^2)}{\tilde{Y}_{22}^2 - A\tilde{Y}_{22} - As^2} \tag{A3}$$

Putting (A3) in (7.35) and simplifying,

$$\Delta_1 = \frac{s^4 \tilde{Y}_{12}^2}{As^2 + A\tilde{Y}_{22} - \tilde{Y}_{22}^2} \tag{A4}$$

Equating (A2) and (A4) and simplifying by putting

$$x = \tilde{Y}_{22} - A/2 \tag{A5}$$

$$y = s\tilde{Y}_{12} \tag{A6}$$

we get

$$y(y - sx) = k - x^2 \tag{A7}$$

where k is given by (7.47). Putting (A1) in (7.34) and rearranging terms, we get

$$s^4(1 + \tilde{Y}_{11}) = \Delta_1(m - x) \tag{A8}$$

where

$$m = s^2 + A/2 \tag{A9}$$

Using (A5) and (A6) in (A2), we get

$$\Delta_1 = \frac{s^2 y}{s - sx} \tag{A10}$$

Putting (A10) in the right-hand side of (A8), we get

$$1 + \tilde{Y}_{11} = \frac{y(m - x)}{s^2(y - sx)} \tag{A11}$$

Split the right-hand side of (7.32) into two parts as

$$A_1 = \tilde{Y}_{11}(\tilde{Y}_{22} + s^2) - \tilde{Y}_{12}^2 \tag{A12}$$

$$A_2 = 2s^2 \tilde{Y}_{12} + s^2 \Delta_2 \tag{A13}$$

From (7.35), (A12) becomes

$$A_1 = \Delta_1 - \tilde{Y}_{22} - s^2 \tag{A14}$$

A_2 may be written as

$$A_2 = 2s^2(\tilde{Y}_{22} + \Delta_2) - \Delta_2 s^2 \tag{A15}$$

Using (7.33) and (7.34) in (A15) and simplifying,

$$A_2 = \Delta_1(2\tilde{Y}_{12} - \tilde{Y}_{22}) \tag{A16}$$

Putting (A14) and (A16) in (7.32) and rearranging,

$$\Delta_1[(1 + \tilde{Y}_{11}) - 2\tilde{Y}_{12} + \tilde{Y}_{22} - A/3 - 2] = -(\tilde{Y}_{22} + s^2) \tag{A17}$$

Using (A5) and (A6) in (A2), we get

$$\Delta_1 = \frac{sy}{\tilde{Y}_{12} - x} \tag{A18}$$

Using (A5), (A6) and (A18) in (A17), we get

$$sy[s(1 + \tilde{Y}_{11}) + xs - 2y + n] = -(x + m)(y - sx) \tag{A19}$$

where

$$n = s(A/6 - 2) \tag{A20}$$

Using (A11) in (A19) and rearranging the terms, we get

$$y^2(m - x) + ys(y - sx)(sx - 2y + n) + (x + m)(y - sx)^2 = 0 \tag{A21}$$

Using (A7) in (A21) and simplifying, we get

$$2m(k - x^2) + sx^2(ms - n) + ks(n + sx) = 2ksy \tag{A22}$$

Treating (A7) as a quadratic in y and solving, we get the value of y as

$$y = \tfrac{1}{2}[sx + \sqrt{s^2x^2 + 4(k - x^2)}] \tag{A23}$$

where the positive root is chosen. Using (A23) in the right hand side of (A22) and simplifying, we get

$$Bx^2 + C = ks\sqrt{x^2(s^2 - 4) + 4K} \tag{A24}$$

where B and C are given by (7.45) and (7.46). Squaring (A24) and putting

$$z = x^2 \tag{A25}$$

we get

$$d_1 z^2 + d_2 z + d_3 = 0 \tag{A26}$$

where d_1, d_2, and d_3 are given by (7.44). Hence from (A26) and (A25), x is obtained as given by (7.43). From (A5), $\tilde{Y}_{22}$ is obtained as given by (7.42). From (A23) and (A24), we get

$$y = \frac{Q_1}{2ks} \tag{A27}$$

where Q_1 is given by (7.48). Hence from (A6), $\tilde{Y}_{12}$ is obtained as given by (7.41). Using (A9) and (A27) in (A11) and simplifying, we get $\tilde{Y}_{11}$ as given by (7.40).

APPENDIX 7B: DERIVATION OF EKSTRAND'S STEADY STATE SOLUTION

Putting all matrices in (6.28), K_f may be obtained as:

$$K_f = \begin{bmatrix} 1 & 0 & 1/\sigma_x^2 & 0 \\ -T & 1 & -T/\sigma_x^2 & 1/\sigma_d^2 \\ -qT^3/6 & qT^2/2 & 1 - qT^3/(6\sigma_x^2) & T + qT^2/(2\sigma_d^2) \\ -qT^2/2 & qT & -qT^2/(2\sigma_x^2) & 1 + qT/\sigma_d^2 \end{bmatrix} \tag{B1}$$

Evaluating (6.37), the characteristic polynomial may be obtained as

$$\lambda^4 - a\lambda^3 + (b+2)\lambda^2 - a\lambda + 1 = 0 \tag{B2}$$

where

$$\begin{aligned} a &= 4 + qT/\sigma_d^2 - qT^3/(6\sigma_x^2) \\ b &= 4 + 2qT/\sigma_d^2 + 2qT^3/(3\sigma_x^2) + q^2T^4/(12\sigma_x^2\sigma_d^2) \end{aligned} \tag{B3}$$

Evaluating (6.40),the eigenvectors may be found as

$$V_i = \begin{bmatrix} 1 \\ d_i \\ e_i \\ f_i \end{bmatrix} \tag{B4}$$

where

$$\begin{aligned} d_i &= \frac{-T\lambda_i(\lambda_i - C)}{D_i} \\ e_i &= (\lambda_i - 1)\sigma_x^2 \\ f_i &= \frac{-qT^2\lambda_i(\lambda_i + 1)}{2D_i} \end{aligned} \tag{B5}$$

where

$$C = 1 + \frac{qT}{2\sigma_d^2}$$

$$D_i = \lambda_i^2 - 2C\lambda_i + 1$$

The steady state $\tilde{P}$ matrix is now given by (6.33) where W_{11} and W_{21} are determined by the eigenvectors as

$$W_{11} = \begin{bmatrix} 1 & 1 \\ d_1 & d_2 \end{bmatrix} \tag{B6}$$

$$W_{21} = \begin{bmatrix} e_1 & e_2 \\ f_1 & f_2 \end{bmatrix}$$

λ_1 and λ_2 are the eigenvalues outside the unit circle. Instead of determining the eigenvalues, it is possible to express $\tilde{P}$ matrix in terms of the sum and product of the eigenvalues as follows:

As the inverse of an eigenvalue is also an eigenvalue, the characteristic equation may also be expressed as

$$\prod_{i=1}^{2}(\lambda - \lambda_i)(\lambda - 1/\lambda_i) = 0 \tag{B7}$$

Expressing (B7) as a polynomial in λ and equating its coefficients with those of (B2), we get

$$\lambda_1 + 1/\lambda_1 + \lambda_2 + 1/\lambda_2 = a \tag{B8}$$

$$(\lambda_1 + 1/\lambda_1)(\lambda_2 + 1/\lambda_2) = b \tag{B9}$$

Equation (B8) can be written as

$$S_1 = aS_2/(1 + S_2) \tag{B10}$$

where

$$S_1 = \lambda_1 + \lambda_2 \tag{B11}$$

$$S_2 = \lambda_1 \lambda_2$$

From (B9) and (B10), we get

$$\left(S_2 + \frac{1}{S_2}\right) + \frac{a^2}{(S_2 + 1/S_2) + 2} = b + 2 \tag{B12}$$

(B12) is a quadratic in $(S_2 + 1/S_2)$ whose solution is given by

$$(S_2 + 1/S_2) = \tfrac{1}{2}[b + \sqrt{(b+4)^2 - 4a^2}] = d \text{ (say)} \tag{B13}$$

From (B13), S_2 may be obtained as

$$S_2 = \tfrac{1}{2}(d + \sqrt{d^2 - 4}) \tag{B14}$$

The positive signs are taken since S_2 is a real number and we require the solution of eigenvalues outside the unit circle. By working out the solution (6.33) using (B10) to (B14), we get the results given by (7.52) to (7.56).

APPENDIX 7C: DERIVATION OF ELEMENTS OF Q MATRIX

The covariance matrix of maneuver noise is given by

$$Q = E\{\omega(\tau)\omega^T(\nu)\} = E\left\{\begin{bmatrix}\omega_1(\tau)\\ \omega_2(\tau)\\ \omega_3(\tau)\end{bmatrix}[\omega_1(\nu) \quad \omega_2(\nu) \quad \omega_3(\nu)]\right\} \tag{C1}$$

where

$$\omega_1(\tau) = \int_0^T \left(\frac{\tau^2}{2}\right)a(\tau)\,d\tau \tag{C2}$$

$$\omega_2(\tau) = \int_0^T \tau a(\tau)\,d\tau$$

$$\omega_3(\tau) = \int_0^T a(\tau)\,d\tau$$

where $a(\tau)$ is the random white noise change in acceleration process. If the Q matrix is defined as given by (6.55), then

$$q_{11} = E\left\{\int_0^T \left(\frac{\tau^2}{2}\right)a(\tau)\,d\tau \int_0^T \left(\frac{\nu^2}{2}\right)a(\nu)\,d\nu\right\} \tag{C3}$$

$$= \int_0^T \int_0^T \left(\frac{\tau^2}{2}\right)\left(\frac{\nu^2}{2}\right)E\{a(\tau)a(\nu)\}\,d\tau\,d\nu$$

For a white noise process,

$$E\{a(\tau)a(\nu)\} = q\delta(\tau - \nu) \tag{C4}$$

where q is the spectral density of the noise and $\delta(x)$ is the Dirac-delta function. Hence (C3) becomes

$$q_{11} = \int_0^T \left(\frac{\nu^4}{4}\right)q\,d\nu = \frac{(q/4)T^5}{5} = \frac{qT^5}{20} \tag{C5}$$

$$q_{12} = \int_0^T \left(\frac{\nu^3}{2}\right)q\,d\nu = \frac{(q/2)T^4}{4} = \frac{qT^4}{8}$$

$$q_{13} = \int_0^T \left(\frac{\nu^2}{2}\right)q\,d\nu = \frac{(q/2)T^3}{3} = \frac{qT^3}{6}$$

$$q_{22} = \int_0^T (\nu^2)q\,d\nu = \frac{(q)T^3}{3} = \frac{qT^3}{3}$$

$$q_{23} = \int_0^T (v)q\,dv = \frac{(q)T^2}{2} = \frac{qT^2}{2}$$

$$q_{33} = \int_0^T q\,dv = (q)T = qT$$

q is equivalent to $\sigma_a^2 T$, where σ_a^2 is the variance of the maneuver noise.

APPENDIX 7D: DETAILS OF DERIVATION OF STEADY STATE RESULTS OF RAMACHANDRA-MOHAN-GEETHA'S MODEL

Putting all matrices in (6.28), K_f may be obtained as:

$$K_f = \begin{bmatrix} 1 & 0 & 0 & 1/\sigma_x^2 & 0 & 0 \\ -\tau\theta & 1 & 0 & -\tau\theta/\sigma_x^2 & 1/\sigma_d^2 & 0 \\ -\tau^2 a_2 & \tau(1-y) & y & -\tau^2 a_2/\sigma_x^2 & \tau(1-y)/\sigma_d^2 & 0 \\ U_1 & S_1 & yq_{13} & 1+U_1/\sigma_x^2 & \tau\theta + yq_{13}/\sigma_d^2 & \tau^2 a_1 \\ U_2 & S_2 & yq_{23} & U_2/\sigma_x^2 & 1+yq_{23}/\sigma_d^2 & \tau(1-e) \\ U_3 & S_3 & yq_{33} & U_3/\sigma_x^2 & yq_{33}/\sigma_d^2 & e \end{bmatrix} \tag{D1}$$

Since our system model is of order 3, K_f is of order 6. If λ is an eigenvalue of K_f, then $1/\lambda$ is also an eigenvalue of K_f and hence the eigenvalue problem is of order 3 only.

Eigenvectors Determination

The eigenvectors corresponding to the eigenvalues λ_i may be obtained by directly evaluating (6.40) as

$$V_i = \begin{bmatrix} 1 \\ d_i \\ e_i \\ f_i \\ g_i \\ h_i \end{bmatrix} \tag{D2}$$

where $i = 1, 2, 3, 4, 5, 6$ and

$$d_i = \frac{-T\lambda_i}{D_i}[\lambda_i^3 + (6\alpha - 1)(3\lambda_i^2 + 1) + 3(16\alpha - 1)\lambda_i] \tag{D3}$$

$$e_i = \frac{T^2\lambda_i}{2D_i}[\lambda_i^3 + (12\alpha - 1)\lambda_i(\lambda_i + 1) + 1]$$

$$f_i = (\lambda_i - 1)\sigma_x^2$$

$$g_i = \frac{qT^4\lambda_i}{24D_i}[\lambda_i^3 + 11\lambda_i(\lambda_i + 1) + 1]$$

$$h_i = \frac{qT^3\lambda_i}{6D_i}[\lambda_i^3 + 3(2\alpha + 1)\lambda_i(\lambda_i - 1) - 1]$$

where

$$D_i = \lambda_i^4 + 4(6\alpha - 1)\lambda_i(\lambda_i^2 + 1) + 6(16\alpha - 1)\lambda_i^2 \tag{D4}$$

with

$$\alpha = qT^3/(6\sigma_d^2) = 24/(rs)^2 \tag{D5}$$

Characteristic Equation

Using (6.37), the characteristic polynomial may be obtained as

$$\lambda^6 - a\lambda^5 + b\lambda^4 - c\lambda^3 + b\lambda^2 - a\lambda + 1 = 0 \tag{D6}$$

where

$$a = 6(1 + 0.2\beta - 4\alpha) \tag{D7}$$
$$b = 3[5 + 16\alpha + 0.8\beta(3\alpha - 13)]$$
$$c = 4[5 + 36\alpha + 1.8\beta(6\alpha + 11)]$$

α, β, and r are given by (7.93), (7.94), and (7.96). s is defined in (7.39).

Since the inverse of an eigenvalue is also an eigenvalue, the characteristic polynomial must be of the form

$$\prod_{i=1}^{3}(\lambda - \lambda_i)(\lambda - 1/\lambda_i) = 0 \tag{D8}$$

where λ_1, λ_2, and λ_3 are the eigenvalues outside the unit circle. Comparing (D1) and (D5), we get after simplification,

$$S_2 + S_1 S_3 = aS_3 \tag{D9}$$
$$S_1 + S_1 S_2 + S_2 S_3 = bS_3 \tag{D10}$$
$$1 + S_1^2 + S_2^2 + S_3^2 = cS_3 \tag{D11}$$

where S_1, S_2, and S_3 are given by

$$S_1 = \lambda_1 + \lambda_2 + \lambda_3 \tag{D12}$$
$$S_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$
$$S_3 = \lambda_1\lambda_2\lambda_3$$

Adding 2 times (D9) to (D11), we get

$$(1 + S_2)^2 + (S_1 + S_3)^2 = (c + 2a)S_3 \tag{D13}$$

Addin S_3 to both sides of (D10), we get

$$(1 + S_2)(S_1 + S_3) = (b + 1)S_3 \tag{D14}$$

Solving (D13) and (D14) simultaneously for $(1 + S_2)$ and $(S_1 + S_3)$, we get

$$S_1 + S_3 = 4d_1 x \tag{D15}$$

$$1 + S_2 = 4d_2 x \tag{D16}$$

where

$$x = \sqrt{S_3} \tag{D17}$$

$$d_1 = \sqrt{w + g} \tag{D18}$$

$$d_2 = \sqrt{w - g}$$

where w and g are defined in (7.91) and (7.92). Using (D17) in (D15)

$$S_1 = 2d_1 x - x^2 \tag{D19}$$

From (D16)

$$S_2 = 4d_2 x - 1 \tag{D20}$$

Using (D17), (D19), and (D20) in (D9), we get the biquadratic

$$x^4 - 4d_1 x^3 + 6e_1 x^2 - 4d_2 x + 1 = 0 \tag{D21}$$

where e_1 is given by (7.98).

When (6.33) is evaluated, it is found that the elements of $\tilde{P}$ matrix may be expressed in terms of the symmetric functions of eigenvalues. These symmetric functions are evaluated in terms of sums and products of S_1, S_2 and S_3 as given separately in Appendix 7E. It is interesting to note that the undetermined factors in eigenvalues cancel out neatly.

After considerable algebraic simplifications, the elements of the $\tilde{P}$ matrix are obtained as given in (7.86).

APPENDIX 7E: VALUES OF SYMMETRIC FUNCTIONS

When (6.33) is evaluated, the elements of the $\tilde{P}$ matrix are found to contain symmetric functions of eigenvalues. These are evaluated and given below. Let

$$F = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)$$

Then the symmetric functions used in (7.82) may be expressed as:

$$\sum \lambda_3\lambda_2(\lambda_3 - \lambda_2) = F$$
$$\sum \lambda_3\lambda_2(\lambda_3^2 - \lambda_2^2) = S_1 F$$
$$\sum \lambda_3\lambda_2(\lambda_3^3 - \lambda_2^3) = (S_1^2 - S_2)F$$
$$\sum \lambda_3^2\lambda_2^2(\lambda_3 - \lambda_2) = S_2 F$$
$$\sum \lambda_3^2\lambda_2^2(\lambda_3^2 - \lambda_2^2) = (S_1 S_2 - S_3)F$$
$$\sum \lambda_3^3\lambda_2^3(\lambda_3 - \lambda_2) = (S_2^2 - S_1 S_3)F$$
$$\sum \lambda_1(\lambda_3^2 - \lambda_2^2) = -F$$
$$\sum \lambda_1(\lambda_3^3 - \lambda_2^3) = -S_1 F$$
$$\sum \lambda_1(\lambda_3^4 - \lambda_2^4) = (S_2 - S_1^2)F$$
$$\sum \lambda_1^2(\lambda_3 - \lambda_2) = F$$
$$\sum \lambda_1^2(\lambda_3^3 - \lambda_2^3) = -S_2 F$$
$$\sum \lambda_1^2(\lambda_3^4 - \lambda_2^4) = (S_3 - S_1 S_2)F$$
$$\sum \lambda_1^3(\lambda_3 - \lambda_2) = S_1 F$$
$$\sum \lambda_1^3(\lambda_3^2 - \lambda_2^2) = S_2 F$$
$$\sum \lambda_1^3(\lambda_3^4 - \lambda_2^4) = (S_1 S_3 - S_2^2)F$$
$$\sum \lambda_1^4(\lambda_3 - \lambda_2) = (S_1^2 - S_2)F$$
$$\sum \lambda_1^4(\lambda_3^2 - \lambda_2^2) = (S_1 S_2 - S_3)F$$
$$\sum \lambda_1^4(\lambda_3^3 - \lambda_2^3) = (S_2^2 - S_1 S_2)F$$
$$\sum \lambda_1^5(\lambda_3 - \lambda_2) = (2S_1 S_2 - S_1^2 - S_3)F$$
$$\sum \lambda_1^5(\lambda_3^2 - \lambda_2^2) = (S_2^2 + S_1 S_3 - S_1^2 S_2)F$$
$$\sum \lambda_1^5(\lambda_3^3 - \lambda_2^3) = (S_2 S_3 + S_1^2 S_3 - S_1 S_2^2)F$$
$$\sum \lambda_1^5(\lambda_3^4 - \lambda_2^4) = (2S_1 S_2 S_3 - S_3^2 - S_2^3)F$$

The factor F will get canceled in the evaluation of (6.33). S_1, S_2, and S_3 are given by (D12).

8

Continuous-Time One-Dimensional Kalman Tracking Filters with Position and Velocity Measurements

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8.1 INTRODUCTION

In this chapter, the random walk velocity (RWV) and the random walk acceleration (RWA) models of continuous-time Kalman tracking filters are discussed. The steady state covariances and gains are obtained analytically for both the models. The position and velocity measurements are assumed to be obtained continuously, and both these measurements are incorporated in the filtering processes of the two models. The filtering solution for a continuous time system may be found in two ways:

1. By a limiting operation on the known solution for the corresponding discrete-time case
2. By directly solving the algebraic Riccati equation

Both methods are demonstrated in this chapter. The results for the corresponding filters in which measurements of one state alone are available are obtained as special cases of these models.

In Ref. 1, Ekstrand discusses the RWV model of a continuous-time Kalman tracking filter. Analytical expressions are given for the steady state solution of the model. The position and velocity measurements are assumed to be obtained continuously, and both these measurements are utilized in the RWV model. Ekstrand obtained the solution by a limiting operation on the known solution for the corresponding discrete-time case [3]. Pachter [2] obtained the solution for the same problem by directly solving the algebraic Riccati equation (ARE). The results for the corresponding filter in which measurements of one state alone are available are obtained by Ekstrand as a special case of this model [1]. These results are shown to be the same as the RWV solution of Fitzgerald [4] and also the solution for the special case $\lambda = 0$ in the ECV model of Nash [5].

Ekstrand [1] demonstrates in his model that if the filter solution is known for a discrete time solution which is obtained by sampling some continuous-time system, then the filtering solution for the continuous-time system can also be found by a limiting operation. The transfer functions of the filter for the case when measurements of two states are available and for the case when measurements of one state only are available are also given by Ekstrand.

In Ref. 6, Ramachandra-Mohan-Geetha's RWA model for a continuous-time Kalman tracking filter is discussed. Steady state covariances and gains are obtained analytically in this model. As in Ekstrand's model the first two states of the filter are assumed to be measured continuously and both these measurements are incorporated in the filtering process. The solution is obtained by directly solving the algebraic Riccati equation. The results for the corresponding filter in which measurements of one state alone are available are obtained as a special case of this model. These results are in excellent agreement with those of Fitzgerald [4], discussed in Section 5.3. The solutions are visualized in two graphs.

8.2 EKSTRAND'S RWV MODEL

Ekstrand's model [1] deals with a continuous-time one-dimensional Kalman tracking filter. It is a two-state RWV model where both states are measured continuously. The steady state results of the continuous-time system are obtained from the known steady state solution to the Kalman filter for the discrete-time system obtained by sampling the continuous RWV system [3]. The continuous-time system is considered as the limit case of the discrete-time system as the sampling time T tends to zero.

8.2.1 Dynamic Model

The dynamic model is represented by the following linear continuous-time constant coefficient system:

$$\dot{X} = FX + U \tag{8.1}$$

where

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{8.2}$$

and

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8.3}$$

If the application is for the development of tracking system as in Fitzgerald's model [4], then x_1 is the position and x_2 is the velocity of the target. The process noise U given by

$$U = \begin{bmatrix} 0 \\ u \end{bmatrix} \tag{8.4}$$

is assumed to be a white noise process with covariance Q given by

$$E\{U(t)U^T(\tau)\} = Q\delta(t - \tau)$$

where

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} \tag{8.5}$$

and δ is the Dirac delta function.

8.2.2 Measurement Model

The measurement model is simply given by

$$Z = X + V \tag{8.6}$$

where

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{8.7}$$

and

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{8.8}$$

z_1 and z_2 are the measurements of the two-state variables x_1 and x_2. V is the white noise measurement process with covariance R given by

$$E\{V(t)V^T(\tau)\} = R\delta(t-\tau)$$

where

$$R = \begin{bmatrix} r_0 & 0 \\ 0 & r_d \end{bmatrix} \tag{8.9}$$

The white noise process U is assumed to be independent of the white noise measurement process V. Further, the position measurement process is also assumed to be independent of the velocity measurement process.

8.2.3 Filtering Equations

The steady state solution to the Kalman filter for this system is determined by the solution to the algebraic Riccati equation

$$FP + PF^T - PR^{-1}P + Q = 0 \tag{8.10}$$

The gain matrix is given by

$$K = PR^{-1} \tag{8.11}$$

Let the covariance matrix be defined as given in (5.11). Then (8.10) gives rise to the following three nonlinear equations:

$$\frac{P_{11}^2}{r_0} + \frac{P_{12}^2}{r_d} = 2P_{12} \tag{8.12}$$

$$P_{12}\left(\frac{P_{11}}{r_0} + \frac{P_{22}}{r_d}\right) = P_{22} \tag{8.13}$$

$$\frac{P_{12}^2}{r_0} + \frac{P_{22}^2}{r_d} = q \tag{8.14}$$

8.2.4 Continuous-Time Filter Solution

The continuous-time filter steady state solution can be obtained by applying the following limiting operation:

$$
\begin{aligned}
T &\to 0 \\
\sigma_x^2 T &\to r_0 \\
\sigma_d^2 T &\to r_d
\end{aligned}
\tag{8.15}
$$

in the continuous discrete-time filter steady state solution. This is the method of handling the transition formally from discrete- to continuous-time systems as given in Refs. 7 and 8.

8.2.5 Steady State Covariances

Consider the case where discrete-time measurements on the continuous-time system are available. This is the continuous-discrete-time case discussed in Chapter 7. Ekstrand's steady state solution for the predicted covariance $\tilde{P}$ is given by (7.52). It may be noted that the predicted covariance $\tilde{P}$ tends to the filtered covariance $\hat{P}$ as the sampling time $T \to 0$. Let this simply be denoted as P. Then applying the limiting operatiom (8.15) in (7.52), the normalized covariances may be obtained as

$$
\begin{aligned}
Y_{11} &= \frac{\sqrt{r(r+\frac{1}{2})}}{1+r} \\
Y_{12} &= \frac{r}{1+r} \\
Y_{22} &= Y_{11}
\end{aligned}
\tag{8.16}
$$

where the normalized dimensionless covariances are given by [1]

$$
\begin{aligned}
Y_{11} &= \frac{P_{11}}{\sqrt{2r_0q_0}} \\
Y_{12} &= \frac{P_{12}}{q_0} \\
Y_{22} &= \frac{P_{22}}{\sqrt{2qq_0}}
\end{aligned}
\tag{8.17}
$$

with

$$q_0 = \sqrt{qr_0} \tag{8.18}$$

$$r = \frac{r_d}{q_0} \tag{8.19}$$

8.2.6 Steady State Gains

If the gain matrix is defined as

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{8.20}$$

then from (8.11), the normalized gains are given by

$$\begin{aligned} G_{11} &= Y_{11} \\ G_{12} &= 1/(1+r) \\ G_{21} &= Y_{12} \\ G_{22} &= Y_{11}/r \end{aligned} \tag{8.21}$$

where the normalized dimensionless gains are defined as [1]

$$\begin{aligned} G_{11} &= K_{11}/\sqrt{2q_1} \\ G_{12} &= K_{12} \\ G_{21} &= \frac{K_{21}}{q_1} \\ G_{22} &= K_{22}/\sqrt{2q_1} \end{aligned} \tag{8.22}$$

where

$$q_1 = \sqrt{q/r_0}$$

8.2.7 Transfer Function of the Filter

The filter equations are given by

$$\dot{\hat{X}} = (F - K)\hat{X} + KZ \tag{8.23}$$

where $\hat{X}$ is an estimate of X and Z is the measurement vector. With F and K defined in (8.2) and (8.20), we get the following transfer function from measurements to estimates:

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \begin{bmatrix} \frac{1}{a}(s+a) & \frac{1}{ar}\frac{1}{b}(s+b) \\ \frac{r}{1+r}s & \frac{1}{1+r}\frac{1}{c}(s+c) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{8.24}$$

where

$$\begin{aligned}\omega_0 &= \sqrt{q_1} \qquad (8.25)\\ \zeta &= (1/\sqrt{2})\sqrt{1+1/2r}\\ a &= [(1+r)/r]\omega_0/2\zeta\\ b &= 2\zeta\omega_0\\ c &= \omega_0/2\zeta\end{aligned}$$

Thus, we have a stable filter as expected from filter theory. It is to be noted that in addition to the two stable poles, there are zeroes which influence the filter performance.

8.2.8 RWV Model with Position Measurements Only

The case with position measurements only is obtained by letting $r_d \to \infty$ in (8.12) to (8.14). When this is done, we obtain

$$\begin{aligned}P_{11}^2/r_0 &= 2P_{12} \qquad (8.26)\\ P_{11}P_{12}/r_0 &= P_{22}\\ P_{12}^2/r_0 &= q\end{aligned}$$

Solving (8.26), the solution may be obtained as

$$\begin{aligned}P_{11} &= \sqrt{2r_0q_0} \qquad (8.27)\\ P_{12} &= q_0\\ P_{22} &= \sqrt{2qq_0}\end{aligned}$$

As $r_d \to \infty$, $r \to \infty$ and we have from (8.21),

$$\begin{aligned}K_{11} &= \sqrt{2q_1} \qquad (8.28)\\ K_{12} &= 0\\ K_{21} &= q_1\\ K_{22} &= 0\end{aligned}$$

(8.27) and (8.28) are the same as the solutions for the special case $\lambda = 0$ in the ECV model given in Ref. 5. They are the same as the RWV solution [4] given by (5.27) to (5.29). The results (8.27) and (8.28) may be obtained directly by putting $r \to \infty$ in (8.16) and (8.21).

8.2.9 Transfer Function of the Filter with Position Measurements Only

For the case with position measurements only, the transfer function from position measurement to estimates is obtained by letting $r \to \infty$ in (8.24). The result is

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \frac{\omega_0^2}{s + 2\zeta\omega_0 s + \omega_0^2} \begin{bmatrix} \frac{1}{a}(s + a) \\ s \end{bmatrix} z_1 \tag{8.29}$$

where

$$\omega_0 = \sqrt{q_1} \tag{8.30}$$

$$\zeta = 1/\sqrt{2}$$

$$a = \omega_0/\sqrt{2}$$

8.2.10 Interpretation of Results

1. In the basic system model, there are three independent parameters r_0, r_d, and q which can influence the solution. However, the normalized solution is expressed as a function of only one parameter r.

2. $Y_{11} = Y_{22} = G_{11}$ and $Y_{12} = G_{21}$ since they have the same solution. Hence the solution is conveniently summarized in two graphs with only two curves in each graph, as illustrated in Figures 8.1 and 8.2. Comparing this with the discrete-time case where more graphs with several curves in each graph were needed, we see that it is easier to get a view of the solution in the continuous-time case. One reason for the increased complexity in the discrete-time case is the addition of one more parameter, the sampling interval T.

3. In (8.17), the solution is normalized by the solution (8.27) which is valid for the case with position measurements only. This choice of normalization enables us to see directly from Figure 8.1 the accuracy improvement obtained by incorporating velocity measurements into the filter. It may be noted that calculated as a percentage, the improvement is the same for the position and velocity estimates.

4. From (8.25) and (8.30), it is seen that the natural resonent frequency ω_0 is independent of the velocity measurement accuracy r_d; thus we get the same value of ω_0 wheather the velocity measurements are used or not.

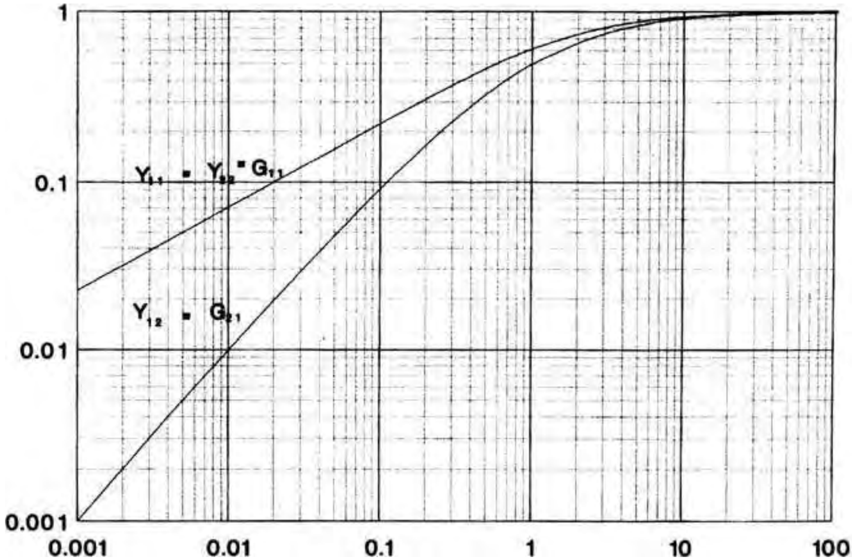

Figure 8.1 Normalized covariances and gains of a function of r. (From Ref. 1, © 1985 IEEE.)

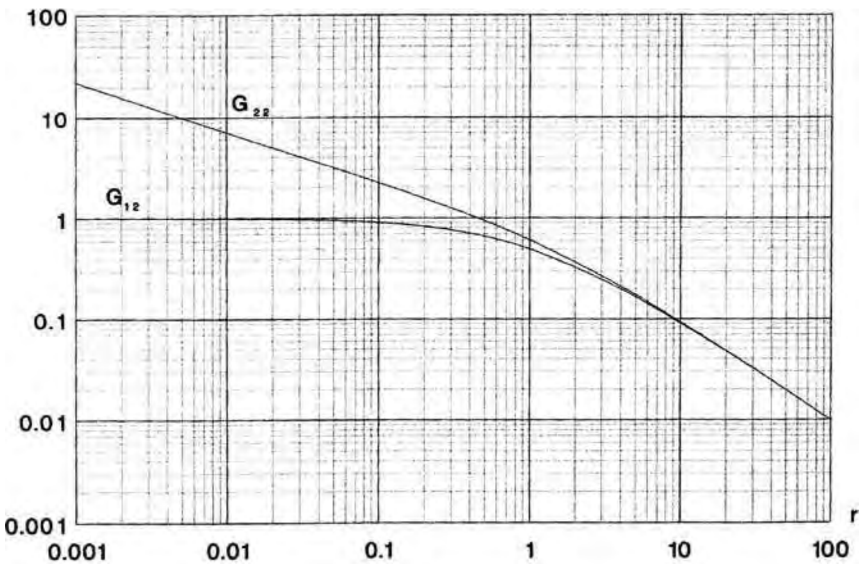

Figure 8.2 Normalized covariances and gains as a function of r. (From Ref. 1, © 1985 IEEE.)

5. From (8.25), it is seen that the damping factor $\zeta \geq 1/\sqrt{2}$, whereas from (8.30), $\zeta = 1/\sqrt{2}$ for the case with position measurements only. Thus, using velocity measurements gives a steady state filter with increased damping factor ζ.

6. The natural frequency ω_0 is the same for the ECV model [5] and for the RWV model with or without velocity measurements.

7. It is easily demonstrated that if the filter solution is known for a discrete time system, then the fiter solution for the continuous-time system can also be found by a limiting operation.

8.3 PACHTER'S STEADY STATE SOLUTION

In Ref. 2, Pachter directly solved the continuous time algebraic Riccati equation (ARE) of Ekstrand's RWV model and obtained the steady state solution.

In (8.12)-(8.14), r_0, r_d, and q are assumed to be greater than zero. Rearranging (8.13), P_{22} may be put as

$$P_{22} = \frac{P_{11}P_{12}}{r_0 y} \tag{8.31}$$

where

$$y = 1 - P_{12}/r_d \tag{8.32}$$

Putting (8.31) into (8.14), we get

$$\frac{P_{12}^2}{r_0}\left[1 + \frac{P_{11}^2}{r_0 r_d y^2}\right] = q \tag{8.33}$$

From (8.12) and (8.32), P_{11}^2 may be written as

$$P_{11}^2 = r_0 P_{12}(1 + y) \tag{8.34}$$

Inserting (8.34) into (8.33), we get

$$\frac{P_{12}^2}{r_0}\left[1 + \frac{P_{12}(1 + y)}{r_d y^2}\right] = q \tag{8.35}$$

From (8.32),

$$P_{12} = r_d(1 - y) \tag{8.36}$$

Putting (8.36) in (8.34), we get

$$P_{11} = \sqrt{r_0 r_d (1 - y^2)} \tag{8.37}$$

Putting (8.36) and (8.37) in (8.31)

$$P_{22} = r_d \sqrt{\frac{r_d}{r_0}} \frac{1-y}{y} \sqrt{1-y^2} \tag{8.38}$$

If we substitute (8.36) into (8.35) and simplify, we obtain the scalar quadratic equation:

$$(1-y)^2 = y^2/r^2 \tag{8.39}$$

where r is defined in (8.19). The two solutions of (8.39) are

$$y_1 = r/(r+1) \tag{8.40}$$

and

$$y_2 = r/(r-1) \tag{8.41}$$

The largest solution of the ARE (in the sense of positive definite matrices) corresponds to the solution $y = y_1$. When $y = y_1$ is inserted into Eqs. (8.36) to (8.38), we get the solutions identical to (8.16).

Two additional real solutions of the ARE (that correspond to $y = y_2$) exist provided $0 < r < \frac{1}{2}$. They are

$$Y_{11} = \pm \frac{\sqrt{r(\frac{1}{2} - r)}}{|r-1|} \tag{8.42}$$

$$Y_{12} = \frac{r}{1-r}$$

$$Y_{22} = Y_{11}$$

8.4 RAMACHANDRA-MOHAN-GEETHA'S MODEL: A THREE-STATE CONTINUOUS-TIME KALMAN TRACKING FILTER

8.4.1 Introduction

In Ref. 4, solutions for the continuous-time Kalman filters for the two-state exponentially correlated velocity model and the three-state exponentially correlated acceleration model are given for the case of position measurements only. Solutions for the special cases of these system models, the so-called random walk velocity model and the random walk acceleration model, are also given in Ref. 4. In these cases, the Kalman filter is based on the continuous measurements of the position state variable only.

In this section, the steady state continuous-time solution is obtained for the three-state random walk acceleration model case where both position and velocity states are measured continuously. Here the solution is obtained by directly solving the algebraic Riccati equation.

8.4.2 Filter Equations

Consider a linear continuous-time constant coefficient system given by (8.1), where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{8.43}$$

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.44}$$

$$U = \begin{bmatrix} 0 \\ 0 \\ u_3 \end{bmatrix} \tag{8.45}$$

x_1 is the position, x_2 is the velocity, and x_3 is the acceleration. The process noise U is assumed to be a white noise process given by

$$E[U(t)U(\tau)^T] = Q\delta(t-\tau)$$

where Q is given by (5.38) and δ is the Dirac delta function.

8.4.3 Measurement Model

The measurement equation is given by

$$Z = HX + E \tag{8.46}$$

where

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{8.47}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{8.48}$$

$$E = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{8.49}$$

z_1 and z_2 are the measurements of x_1 and x_2, respectively. The covariance of the measurement noise process is given by

$$E[E(t)E(\tau)^T] = R\delta(t - \tau)$$

where

$$R = \begin{bmatrix} r_0 & 0 \\ 0 & r_d \end{bmatrix} \tag{8.50}$$

Thus the position measurement process is assumed to be independent of the velocity measurement process. Also, the process noise U is independent of the of the white noise measurement process E.

8.4.4 Filtering Equations

The steady state solution to the Kalman filter for this system is determined by the solution to the algebraic Riccati equation:

$$FP + PF^T - PH^T R^{-1} HP + Q = 0 \tag{8.51}$$

The gain matrix is given by

$$K = PH^T R^{-1} \tag{8.52}$$

If the covariance matrix P is defined as (5.42), then from (8.51), we get the following six nonlinear equations:

$$\frac{P_{11}^2}{r_0} + \frac{P_{12}^2}{r_d} = 2P_{12} \tag{8.53}$$

$$\frac{P_{12}^2}{r_0} + \frac{P_{22}^2}{r_d} = 2P_{23}$$

$$\frac{P_{13}^2}{r_0} + \frac{P_{23}^2}{r_d} = q$$

$$\frac{P_{11}P_{12}}{r_0} + \frac{P_{12}P_{22}}{r_d} = P_{13} + P_{22}$$

$$\frac{P_{11}P_{13}}{r_0} + \frac{P_{12}P_{23}}{r_d} = P_{23}$$

$$\frac{P_{12}P_{13}}{r_0} + \frac{P_{22}P_{23}}{r_d} = P_{33}$$

Let the normalized dimensionless covariances be defined as

$$Y_{11} = \frac{P_{11}}{r_0(q/r_0)^{1/6}} \tag{8.54}$$
$$Y_{12} = \frac{P_{12}}{r_0(q/r_0)^{1/3}}$$
$$Y_{13} = \frac{P_{13}}{r_0(q/r_0)^{1/2}}$$
$$Y_{22} = \frac{P_{22}}{r_0(q/r_0)^{1/2}}$$
$$Y_{23} = \frac{P_{23}}{r_0(q/r_0)^{2/3}}$$
$$Y_{33} = \frac{P_{33}}{r_0(q/r_0)^{5/6}}$$

Then the six nonlinear equations (8.53) may be expressed in terms of normalized quantities (8.54) as

$$Y_{11}^2 + \frac{Y_{12}^2}{r} = 2Y_{12} \tag{8.55}$$
$$Y_{12}^2 + \frac{Y_{22}^2}{r} = 2Y_{23} \tag{8.56}$$
$$Y_{13}^2 + \frac{Y_{23}^2}{r} = 1 \tag{8.57}$$
$$Y_{11}Y_{12} + \frac{Y_{12}Y_{22}}{r} = Y_{13} + Y_{22} \tag{8.58}$$
$$Y_{11}Y_{13} + \frac{Y_{12}Y_{23}}{r} = Y_{23} \tag{8.59}$$
$$Y_{12}Y_{13} + \frac{Y_{22}Y_{23}}{r} = Y_{33} \tag{8.60}$$

where

$$r = \frac{r_d}{(qr_0^2)^{1/3}} \tag{8.61}$$

Solving Eqs. (8.55) to (8.60), we get

$$Y_{11} = yg/h \tag{8.62}$$
$$Y_{12} = y^2/h$$
$$Y_{13} = 1/h$$
$$Y_{22} = (y^3g/h) - 1$$

$$Y_{23} = Y_{11}$$

$$Y_{33} = y^2 - yg/(rh)$$

where

$$g = \sqrt{2 + y^2/r} \tag{8.63}$$

$$h = 1 + y^2/r$$
$$y = (\sqrt{u} + \sqrt{v})/2$$
$$v = 2\sqrt{u^2 + 4/r} - u$$
$$u = A + B$$
$$A = [4(1 + f)]^{1/3}$$
$$B = [4(1 - f)]^{1/3}$$
$$f = \sqrt{1 + 4/(3r)^3}$$

Thus all the normalized covariances are expressed as functions of a single parameter r defined in (8.61).

8.4.5 Steady State Gain Matrix

If the gain matrix is defined as

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \\ K_{31} & K_{32} \end{bmatrix} \tag{8.64}$$

then we have

$$K_{11} = P_{11}/r_0 \tag{8.65}$$

$$K_{21} = P_{12}/r_0$$

$$K_{31} = P_{13}/r_0$$

$$K_{12} = P_{12}/r_d$$

$$K_{22} = P_{22}/r_d$$

$$K_{32} = P_{23}/r_d$$

If the normalized gains are defined as

$$
\begin{aligned}
G_{11} &= \frac{K_{11}}{(q/r_0)^{1/6}} \\
G_{21} &= \frac{K_{21}}{(q/r_0)^{1/3}} \\
G_{31} &= \frac{K_{31}}{(q/r_0)^{1/2}} \\
G_{12} &= K_{12} \\
G_{22} &= \frac{K_{22}}{(q/r_0)^{1/6}} \\
G_{32} &= \frac{K_{32}}{(q/r_0)^{1/3}}
\end{aligned}
\tag{8.66}
$$

then they may be derived as

$$
\begin{aligned}
G_{11} &= Y_{11} \\
G_{21} &= Y_{12} \\
G_{31} &= Y_{13} \\
G_{12} &= Y_{12}/r \\
G_{21} &= Y_{22}/r \\
G_{32} &= Y_{23}/r
\end{aligned}
\tag{8.67}
$$

Thus the normalized covariances and gains are all expressed in terms of a single parameter r and hence are plotted against this parameter in Figures 8.3 and 8.4. We also note that

$$
\begin{aligned}
Y_{11} &= Y_{23} = G_{11} \\
Y_{12} &= G_{21} \\
Y_{13} &= G_{31}
\end{aligned}
\tag{8.68}
$$

as the normalized solution is the same in respect of these covariances and gains.

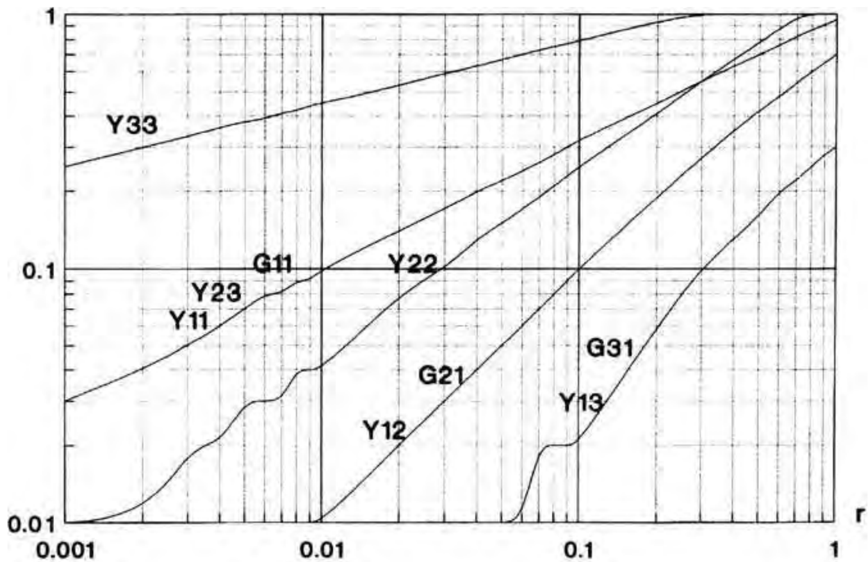

Figure 8.3 Normalized covariances and gains as functions of r.

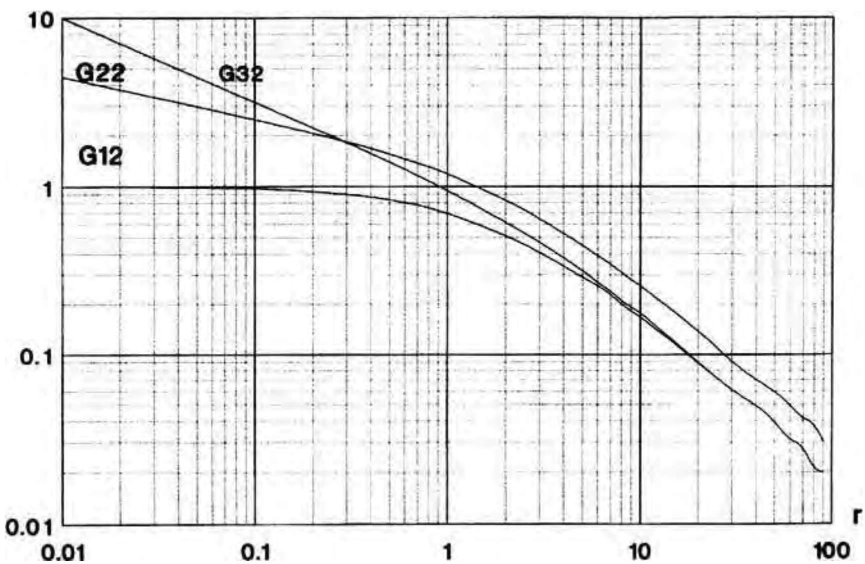

Figure 8.4 Normalized gains as functions of r.

8.4.6 The Case with Position Measurements Only

The case with position measurements only is obtained by letting r tend to infinity. For this case we obtain from (8.63):

$$
\begin{aligned}
f &= 1 \\
B &= 0 \\
A &= u = v = 2 \\
y &= g = \sqrt{2}
\end{aligned}
\tag{8.69}
$$

and

$$h = 1 \tag{8.69}$$

From (8.62) we get

$$
\begin{aligned}
Y_{11} &= 2 \\
Y_{12} &= 2 \\
Y_{13} &= 1 \\
Y_{22} &= 3 \\
Y_{23} &= 2
\end{aligned}
\tag{8.70}
$$

and

$$Y_{33} = 2 \tag{8.70}$$

and from (8.54) we get

$$
\begin{aligned}
P_{11} &= 2r_0(q/r_0)^{1/6} \\
P_{12} &= 2r_0(q/r_0)^{1/3} \\
P_{13} &= r_0(q/r_0)^{1/2} \\
P_{22} &= 3r_0(q/r_0)^{1/2} \\
P_{23} &= 2r_0(q/r_0)^{2/3} \\
P_{33} &= 2r_0(q/r_0)^{5/6}
\end{aligned}
\tag{8.71}
$$

and from (8.66),

$$
\begin{aligned}
K_{11} &= 2(q/r_0)^{1/6} \\
K_{21} &= 2(q/r_0)^{1/3} \\
K_{31} &= (q/r_0)^{1/2}
\end{aligned}
\tag{8.72}
$$

and $K_{12} = K_{22} = K_{32} = 0$ as expected. The solutions (8.71) and (8.72) are in perfect agreement with the solutions given in Ref. 4 for the case of random walk acceleration model as given in equations (5.73) and (5.74).

The details of solving the six nonlinear equations (8.55) to (8.60) are given separately in Appendix 8B.

8.5 SUMMARY

Ekstrand's RWV model [1] of a continuous-time Kalman tracking filter is discussed in Section 8.2. Analytical expressions are given for the steady state solution of the model. The position and velocity measurements are assumed to be obtained continuously and both these measurements are utilized in Ekstrand's model. Ekstrand obtained the solution by a limiting operation on the known solution for the corresponding discrete-time case [3]. The results for the corresponding filter in which measurements of one state alone are available are obtained by Ekstrand as a special case of this model [1]. These results are shown to be the same as the RWV solution of Fitzgerald [4] and also the solution for the special case $\lambda = 0$ in the ECV model of Nash [5]. The transfer functions of the filter for both the cases are given. In Section 8.3, the solution obtained by Pachter [2] by directly solving the algebraic Riccati equation is given. A continuous-time three-state Kalman filter in which two states are measured is discussed in Section 8.4. The covariances and gains are analytically determined by directly solving the algebraic Riccati equation and are expressed as functions of a single parameter. The results for the case when only measurements of one state variable are available are obtained as a special case of this model and these results are in excellent agreement with the results of the random walk acceleration model case of Fitzgerald [4]. The solutions are visualized in two graphs.

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APPENDIX 8A: DERIVATION OF STEADY STATE RESULTS BASED ON LIMITING OPERATION

In Ref. 1, Ekstrand demonstrates the application of limiting operation given in (8.15) to obtain the steady state results. From (7.52) and (7.31), $\tilde{Y}_{12}$ is given by

$$\tilde{Y}_{12} = \frac{\tilde{P}_{12}}{\sigma_x^2/T} = \frac{4}{r^2}[\sqrt{\alpha + r^2} + \sqrt{\alpha}]\beta \tag{A1}$$

Making a slight rearrangement, we may write (A1) as

$$\tilde{P}_{12} = \sigma_x^2 T \frac{4}{rT^2}\left(\sqrt{\frac{1}{r}\frac{\alpha}{r} + 1} + \sqrt{\frac{1}{r}\frac{\alpha}{r}}\right)\beta \tag{A2}$$

where r and s are given by

$$r = \frac{4\sigma_x}{T\sqrt{qT}} \tag{A3}$$

$$s = \frac{\sigma_d T}{\sigma_x}$$

or

$$\frac{1}{r} = \frac{T^2}{4}\sqrt{\frac{q}{\sigma_x^2 T}} \to 0 \tag{A4}$$

$$\frac{4}{rT^2} = \sqrt{\frac{q}{\sigma_x^2 T}} \to \sqrt{\frac{q}{r_0}} \tag{A5}$$

Now consider the factor α/r:

$$\frac{\alpha}{r} = \frac{\alpha_1}{r} + \frac{\alpha_2}{r} \tag{A6}$$

where

$$\frac{\alpha_1}{r} = \frac{4}{3r} + \frac{4}{rs^2} + \frac{8}{3(rs)^2}\frac{1}{r} \tag{A7}$$

$$\frac{\alpha_2}{r} = 2\sqrt{\left(1 + \frac{1}{3r^2}\right)\left[1 + \frac{4}{(rs)^2}\right]\left[1 + \frac{4}{3(rs)^2}\right]} \tag{A8}$$

we have

$$\frac{1}{rs} = \frac{T}{4}\sqrt{\frac{q}{\sigma_d^2 T}} \to 0 \tag{A9}$$

$$\frac{4}{rs^2} = \frac{\sqrt{q\sigma_x^2 T}}{\sigma_d^2 T} \to \frac{q_0}{r_d} \tag{A10}$$

where from (8.18),

$$q_0 = \sqrt{qr_0} \tag{A11}$$

From (A4), (A9), and (A10), (A7) and (A8) become

$$\frac{\alpha_1}{r} \to \frac{q_0}{r_d} \tag{A12}$$

$$\frac{\alpha_2}{r} \to 2 \tag{A13}$$

and hence, from (A6)

$$\frac{\alpha}{r} \to \frac{q_0}{r_d} + 2 \tag{A14}$$

From (7.54), β is given by

$$\beta = \sqrt{\frac{1 + 4/(rs)^2}{1 + [4/(rs^2)](\alpha/r)}} \to \frac{1}{1 + 1/r} \tag{A15}$$

From (A2),

$$\tilde{P}_{12} \to \frac{q_0}{1 + 1/r} = P_{12} \tag{A16}$$

or

$$Y_{12} = \frac{r}{1 + r} \tag{A17}$$

which is the solution given in (8.16) for Y_{12}. Similarly, the solutions for Y_{11} and Y_{22} may be obtained in a straightforward manner.

APPENDIX 8B: SOLUTION OF NONLINEAR EQUATIONS

The method of solving equations (8.55) to (8.60) is briefly given below:
From (8.59) we obtain

$$Y_{11}^2 Y_{13}^2 = Y_{23}^2(1 - Y_{12}/r)^2 \tag{B1}$$

Putting the values of Y_{11}^2 from (8.55) and Y_{23}^2 from (8.57) in (B1) and simplifying, we get

$$Y_{13} = 1 - Y_{12}/r \tag{B2}$$

Putting (B2) in (B1),we get

$$Y_{11} = Y_{23} \tag{B3}$$

Using (B2) and (B3) in (8.57), we get

$$Y_{11}^2 = r(1 - Y_{13}^2) \tag{B4}$$

From (8.58) and (B2), we get

$$(Y_{11}Y_{12} - Y_{13})^2 = Y_{22}^2 Y_{13}^2 \tag{B5}$$

From (8.56) and (B3),

$$Y_{22}^2 = r(2Y_{11} - Y_{12}^2) \tag{B6}$$

Using (B2), (B4), and (B6) in (B5) and simplifying we get

$$[r^2(1 - Y_{13})^2 + Y_{13}^2]^2 = 4r^3(1 - Y_{13}^2)Y_{13}^2 \tag{B7}$$

Dividing (B7) throughtout by $r^6 Y_{13}^4$ and putting

$$w = \frac{1}{Y_{13}} - 1 \tag{B8}$$

and simplifying, we get

$$r^2w^2 - 2\sqrt{2rw} - 1/r = 0 \tag{B9}$$

Putting

$$Y = \sqrt{rw} \tag{B10}$$

in (B9), we get

$$Y^4 - 2\sqrt{2}Y + 1/r = 0 \tag{B11}$$

Solving this biquadratic (B11), we get the value of Y as given in (8.63). Knowing Y, w is obtained from (B10) and hence Y_{13} is obtained from (B8). Using Y_{13} in (B2), Y_{12} is obtained. Using Y_{12}, Y_{11} is obtained from (8.55) and this is also equal to Y_{23} as given in (B3).

Using the values of Y_{11}, Y_{12}, and Y_{13}, Y_{22} and Y_{33} are obtained from (8.58) and (8.60), respectively. Thus the complete solution (8.62) is obtained in this way as a function of a single parameter r given by (8.61).

9
Maneuvering Target Tracking

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9.1 INTRODUCTION

In designing tracking filters for civil and defense applications, a maneuvering aircraft can be modeled by a linear system with random noise accelerations, as discussed in earlier chapters. The trackers provide optimum estimates of the aircraft's position and velocity provided the dynamic model on which the filter is based is a correct representation of the actual nature of flight path. Models based on the assumption that the aircraft flies a constant velocity, straight-line trajectory will eventually lose track if the aircraft deviates from the type of flight path.

Maneuvering targets are well modeled by Singer [1] assuming a linear acceleration model driven by random noise with variance chosen according to a distribution of the potential maneuver accelerations. This filter not only maintains track through the maneuver but also provides good estimates of position, velocity, and acceleration if the maneuver parameter is correctly chosen. If the aircraft is not maneuvering, then there will be a degradation in the performance of the filter compared to simpler filters based on the constant velocity straight-line motion.

Hence some sort of an adaptivity should be built into the tracker so that a more general algorithm is used only when the aircraft is maneuvering. Usually a statistical decision test is applied to detect a maneuver [2]. As long as no maneuver is detected, a simpler filter based on a constant velocity model is used for tracking an aircraft. When a maneuver is detected, the tracker is reinitialized using stored data. Algorithms incorporating such an adaptivity are called maneuver detectors. In its simplest form, usually two Kalman filters are used, one appropriate for the constant velocity motion and the other more appropriate for tracking maneuvering targets. The decision as to which filter is to be used depends upon the value of a test statistic related to a measurement residual. If the test statistic exceeds a certain threshold, then a maneuver is declared and the filter appropriate for the maneuvering target is used. The general theory for tracking maneuvering targets is given in Ref. 2. Several models [4–31] deal with the problem of tracking maneuvering targets based on Kalman filter theory. The performance evaluation of some of these models is given in Ref. 3.

In this chapter, the following two models for maneuvering target tracking are discussed.

1. Bar-Shalom–Birmiwal's model
2. Blom-Bar–Shalom's interacting multiple model

Bar-Shalom and Birmiwal [24] proposed a tracking scheme which will guarantee optimum performance for both nonmaneuvering and maneuvering portions of the trajectory. This scheme consists of a quiescent two-state constant velocity model for nonmaneuvering targets, a maneuver following logic, and a three-state constant acceleration model for the maneuvering trajectories. Once a maneuver is detected, it is assumed that the actual maneuver started a few measurements earlier. Then using the stored measurements, the higher-order acceleration filter is initiated. The filter will be cycled through the stored measurements to reach the current data point. Then the acceleration filter will run in real time with the arrival of new measurements. Now an end-of-maneuver detector will monitor the estimated accelerations. Once the acceleration estimate becomes statistically insignificant, an end-of-maneuver is declared and the constant velocity filter takes over. Thus the switching between the velocity and acceleration filters will take place depending on whether the target is maneuvering or not.

The main disadvantage of this algorithm is that model switching always happens with a time lag due to maneuver detection. During this period, tracking errors increase nonlinearly and may not be acceptable in many applications.

In the multiple-model approach [25], several models for target dynamics are postulated. A filter is set up for each and based on their likelihood functions, the probability for each model being the correct representative of the target dynamics is computed. The state estimate is the weighted average of the model-conditioned estimates with the computed probabilities as weights. But, in practice, the model estimate with the highest probability may be taken as the final estimate. One advantage of this approach is that there is no maneuver following scheme and the probabilities will get adjusted automatically. But the computational complexity of this approach is more compared to Bar-Shalom–Birmiwal's model.

Blom and Bar-Shalom [26] proposed a scheme which is sequel to the multilpe-model approach. This algorithm is called the interacting multiple model (IMM) algorithm. IMM also uses a bank of filters. But the filters, instead of working independently, interact with each other in a probabilistic manner. Due to this interaction, individual filters could adjust their parameters and provide optimum output corresponding to the input. For the purpose of system output, a weighted average of the individual filter outputs is taken [26–31]. The weighting factors are available as part of the filter formulation. Also, there is no need for a separate maneuver detector as in the case of Bar-Shalom–Birmiwal's model. In Ref. 31, Mazor, Averbuch, Bar-Sholom, and Dayan give an exhaustive survey of IMM methods in target tracking.

9.2 BAR-SHALOM–BIRMIWAL'S MODEL

In this section, Bar-Shalom–Birmiwal's model [24] for tracking a maneuvering target is discussed. In this approach, two Kalman filters of different dimensions are used. A constant velocity model is used when the target is not maneuvering, and a constant acceleration model is used during a maneuver.

9.2.1 Dynamic Models

In the absence of maneuver, the target dynamics is modeled as [24]

$$X(k+1) = FX(k) + GW(k) \tag{9.1}$$

where

$$X = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} T/2 & 0 \\ 1 & 0 \\ 0 & T/2 \\ 0 & 1 \end{bmatrix}$$

and

$$W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

The statistical properties of the process noise are

$$E\{W(k)\} = 0$$

$$E\{W(k)W^T(j)\} = Q\delta_{kj}$$

Let the initial state estimate be $\hat{X}(0|0)$ with covariance $\hat{P}(0|0)$.

In the presence of a maneuver, the target dynamics is modeled as [24]

$$X^m(k+1) = F^m X^m(k) + G^m W^m(k) \tag{9.2}$$

where

$$X^m = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \end{bmatrix}$$

$$F^m = \begin{bmatrix} 1 & T & 0 & 0 & T^2/2 & 0 \\ 0 & 1 & 0 & 0 & T & 0 \\ 0 & 0 & 1 & T & 0 & T^2/2 \\ 0 & 0 & 0 & 1 & 0 & T \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G^m = \begin{bmatrix} T^2/4 & 0 \\ T/2 & 0 \\ 0 & T^2/4 \\ 0 & T/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$W^m = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

The statistical properties of the process noise are

$$E\{W^m(k)\} = 0$$
$$E\{W^m(k)W^{mT}(j)\} = Q^m\delta_{kj}$$

The algorithm is not restricted to the above two models only. Any other suitable models may be employed.

9.2.2 Measurement Models

In the absence of maneuver, the measurement equation is given by

$$Z(k) = HX(k) + V(k) \tag{9.3}$$

where

$$Z(k) = \begin{bmatrix} x_m(k) \\ y_m(k) \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$V(k) = \begin{bmatrix} v_x(k) \\ v_y(k) \end{bmatrix}$$

The statistical properties of the measurement noise are

$$E\{V(k)\} = 0$$

and

$$E\{V(k)V^T(j) = R\delta_{kj}$$

In the presence of maneuver, the measurement equation is given by

$$Z(k) = H^m X^m(k) + V(k)$$

with

$$H^m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

9.2.3 Maneuver Detector

A detection statistic that determines whether a maneuver has occurred is developed as follows. Let the matrix δ_k be defined as

$$\delta_k = v^T(k)S^{-1}(k)v(k) \tag{9.4}$$

where $v(k)$ is the measurement residual with covariance S_k. A "fading memory" average of the innovations is constructed as

$$\mu(k) = \alpha\mu(k-1) + \delta(k) \tag{9.5}$$

where

$$\mu(0) = 0$$

and

$$0 \leq \alpha < 1$$

The quantity

$$\Delta = 1/(1-\alpha) \tag{9.6}$$

may be considered as an effective window length over which the presence of a maneuver is detected. A maneuver is declared if at time k, it is found that

$$|\mu(k)| \geq \lambda \tag{9.7}$$

where λ is some chosen threshold. Then the estimator switches from the constant velocity model to the maneuvering model.

When using the constant acceleration filter, at each time point k the test statistic

$$\delta_\alpha(k) = \hat{a}^T(k|k)[P_\alpha^m(k|k)]^{-1}\hat{a}(k|k) \tag{9.8}$$

is calculated where $\hat{a}(k|k)$ is the estimate of the acceleration and $P_{\alpha}^{m}(k|k)$ is the corresponding block from the error covariance matrix. Let p be a window length in time. If the quantity

$$\mu_{\alpha}(k) = \sum_{j=k-p+1}^{k} \delta_{\alpha}(j) \tag{9.9}$$

falls below some chosen threshold, the maneuver is deemed to have ended and the filter switches back to the constant velocity filter.

9.2.4 State Estimation

When a maneuver is detected at time k, the filter is initialized by assuming that the maneuver occurred at some time point $(k - \Delta - 1)$. Let

$$n = k - \Delta$$

$$m = n - 1$$

Δ is the effective window length given by (9.6). The state estimates within the window are then modified as follows.

The estimates of the acceleration components at n are given by

$$\hat{X}_{4+i}^{m}(n|n) = \frac{2}{T^2}[Z_i(n) - Z_i(n|m)] \qquad i = 1, 2 \tag{9.10}$$

The estimates of the position components at n are given by

$$\hat{X}_{2i-1}^{m}(n|n) = Z_i(n) \qquad i = 1, 2 \tag{9.11}$$

The estimates of the velocity components are corrected with the acceleration estimates as

$$\hat{X}_{2i}^{m}(n|n) = \hat{X}_{2i}(m|m) + T\hat{X}_{4+i}^{m}(n|n) \qquad i = 1, 2 \tag{9.12}$$

Let the covariance matrix associated with the modified state estimates as given by (9.10) to (9.12) be denoted as $P^m(n|n)$. Then the elements of this matrix are derived as [24]

$$\begin{aligned}
P^m_{11}(n|n) &= R_{11} \\
P^m_{12}(n|n) &= \frac{2}{T}R_{11} \\
P^m_{15}(n|n) &= \frac{2}{T^2}R_{11} \\
P^m_{22}(n|n) &= \frac{4}{T^2}[R_{11} + P_{11}(m|m)] + \frac{4}{T}P_{12}(m|m) + P_{22}(m|m) \\
P^m_{25}(n|n) &= \frac{4}{T^3}[R_{11} + P_{11}(m|m)] + \frac{6}{T^2}P_{12}(m|m) + \frac{2}{T}P_{22}(m|m) \\
P^m_{55}(n|n) &= \frac{4}{T^2}[R_{11} + P_{11}(m|m) + 2TP_{12}(m|m) + T^2P_{22}(m|m)]
\end{aligned} \tag{9.13}$$

In this model, it is assumed that x and y coordinates are independent. x, $\dot{x}$, and $\ddot{x}$ correspond to components 1, 2, and 5 of the state vector X^m. The covariance matrix elements corresponding to these x components are given by (9.13). Similarly, the covariance matrix corresponding to y, $\dot{y}$, and $\ddot{y}$ are components 3, 4, and 6 of the state vector and may be obtained as

$$\begin{aligned}
P^m_{33}(n|n) &= R_{22} \\
P^m_{34}(n|n) &= \frac{2}{T}R_{22} \\
P^m_{36}(n|n) &= \frac{2}{T^2}R_{22} \\
P^m_{44}(n|n) &= \frac{4}{T^2}[R_{22} + P_{33}(m|m)] + \frac{4}{T}P_{34}(m|m) + P_{44}(m|m) \\
P^m_{46}(n|n) &= \frac{4}{T^3}[R_{22} + P_{33}(m|m)] + \frac{6}{T^2}P_{34}(m|m) + \frac{2}{T}P_{44}(m|m) \\
P_{66}(n|n) &= \frac{4}{T^2}[R_{22} + 2TP_{34}(m|m) + P_{33}(m|m)]
\end{aligned} \tag{9.14}$$

As the model does not require the a priori knowledge of the maneuvering characteristics of the target, it may be regarded as nonparametric. Bar-Shalom and Birmiwal have demonstrated the effectiveness of the algorithm in tracking some typical maneuvers through computer simulation. Also, through a rigorous statistical analysis, it is shown that significant performance improvement is provided by this algorithm when compared with the input estimation algorithm of Chan [11].

9.3 BLOM–BAR-SHALOM'S INTERACTING MULTIPLE MODEL (IMM)

9.3.1 Introduction

The IMM estimator is a suboptimal hybrid filter. This estimator has the ability to estimate the state of a dynamic system with several behavior modes which can switch from one to another. This can be considered to be a self-adjusting variable bandwidth filter and hence very well suited for tracking maneuvering targets [26–31].

The hybrid systems are characterized by the following.

1. State (consisting of kinematic components and possibly feature components also) that evolves according to a stochastic difference (or differential) equation model.
2. Model that is governed by a discrete stochastic process: It is one of a finite number of possible models (each corresponding to a behavior mode) that undergoes jumps (switches) from one model (behaviour mode) to another according to a set of transition probabilities.

The highlight of hybrid models for tracking algorithms is that the occurance of target maneuvers can be explicitly included in the kinamatic equations through regime jumps. The multiple-model adaptive estimation approach is based on the fact that the behavior of the target cannot be characterized by a single model, but a finite number of models can adequately describe its behavior in different regimes.

9.3.2 Design Parameters of an IMM Algorithm

There are three design parameters that characterize an IMM algorithm [26–31]:

1. The set of models for various regimes and their structure.
2. The process noise intensities for various models, in particular the nonmaneuvering model with low-level process noise and the maneuvering model(s) with certain higher noise levels, determined by the assumed maneuverability of the targets.
3. The jump structure (usually Markov) and the transition probabilities between the models from the selected sets. The probabilities are chosen according to the designer's belief about the frequency of the regime switches and can be subsequently adjusted based on Monte Carlo simulation results.

9.3.3 Properties of an IMM Algorithm

The IMM algorithm has the following three desirable properties:

1. It is recursive.
2. It is modular.
3. It has fixed computation requirements per cycle.

9.3.4 Three Major Steps of an IMM Algorithm in Each Cycle

In each cycle, the IMM algorithm consists of the following three major steps:

Step 1: Interaction/Mixing
Step 2: Filtering
Step 3: Combination

9.3.5 Interaction

At each time, the initial condition of the filter matched to a certain mode (a module) is obtained by mixing the state estimates of all filters at the previous time under the assumption that this particular mode is in effect at the current time.

9.3.6 Filtering

The interaction is followed by a regular filtering (prediction and update) step performed in parallel for each mode.

9.3.7 Combination

A combination/weighted sum of the updated state estimates of all filters yields the state estimate. The probability of a mode being in effect plays a key role in the weighting of the mixing and the combination of states and covariances.

9.3.8 IMM Algorithm

A jump linear fixed structure hybrid system with mode transition modeled by a semi-Markov process can be described by the equations given by Refs 26 to 31:

Dynamic model: $$X_{k+1} = F_j(k)X_k + \Gamma_j(k)v_j(k) \tag{9.15}$$

Measurement model: $$Z_k = H_j(k)X_k + W_j(k) \tag{9.16}$$

Equations (9.15) and (9.16) represent the simplest hybrid system with mode transition governed by a first-order homogeneous Markov chain given by

$$P_{i,j}\{m_j(k+1)|m_i(k)\} = p_{ij} \tag{9.17}$$

where p_{ij} is the Markov transition probability from mode i to mode j.

$$m_j(k) = \{m(k) = j\} \tag{9.18}$$

is the event that mode j is in effect at time k. $m(k)$ is the modal state (system mode index) at time k, which denotes the mode in effect during the sampling period ending at k.

In (9.15) and (9.16), it is assumed that the process and measurement noises are gaussian mutually uncorrelated with zero mean and known covariances Q_j and R_j, respectively.

Each mode-matched filter is a standard Kalman filter.

In a fixed structure (or fixed mode set) hybrid system, a set of mode must be selected in advance.

The switching process considered is of the semi-Markov type. The process is specified by a family of transition matrices $p_{ij}(\tau_i)$ where τ_i is the sojourn time of the system in model i. The current probabilities of transition are defined as

$$p_{ij}(\tau_i) = P\{m_j(k+1)|m_i(k), \tau_i(k) = \tau\} \tag{9.19}$$

$\tau_i(k)$ is the sojourn time in state i at time k. For $k = 0$, $\tau = 1$. Thus the values of τ are taken from 1 to the maximum, which at time k is then $k + 1$.

The IMM algorithm basically consists of a group of r filters which run in parallel, and a global computation process collects the results of the filters and produces output estimation. One cycle of the IMM algorithm consists of the following steps:

9.3.8.1 Interaction/Mixing

The mixing probability at time $k-1$ (the weights with which the estimates from the previous cycle are given to each filter at the beginning of the current cycle) is given by

$$\mu_{i|j}(k-1|k-1) = \frac{1}{\bar{c}_j} p_{ij}\mu_i(k-1) \tag{9.20}$$

where $i, j = 1, 2, \ldots, r$, $\mu_i(k-1)$ is the mode probability at time $k-1$ and $\bar{c}_j$

is the normalization factor given by

$$\bar{c}_j = \sum_{i=1}^{r} p_{ij}\mu_i(k-1) \tag{9.21}$$

$j = 1, 2, \ldots, r$.

The mixed initial condition of the state estimate for mode matched filter j at time $k-1$ is given by

$$\hat{X}_{0j}(k-1|k-1) = \sum_{i=1}^{r} \hat{X}_i(k-1|k-1)\mu_{i|j}(k-1|k-1) \tag{9.22}$$

where $\mu_{i|j}(k-1|k-1)$ is given by (9.20). The covariance corresponding to the estimate (9.22) is given by

$$P_{0j}(k-1|k-1) = \sum_{i=1}^{r} \{P_i(k-1|k-1) + A_j A_j^T\}\mu_{i|j}(k-1|k-1) \tag{9.23}$$

where

$$A_j = \hat{X}_i(k-1|k-1) - \hat{X}_{0j}(k-1|k-1) \tag{9.24}$$

$j = 1, 2, \ldots, r$. The estimate (9.22) and covariance (9.23) are used as input to the mode matched Kalman filter j.

9.3.8.2 Filtering

The optimal predicted estimate of the state vector in mode matched filter j is given by

$$\hat{X}_j(k|k-1) = F_j(k-1)\hat{X}_{0j}(k-1|k-1) \tag{9.25}$$

The predicted covariance matrix of estimation errors in mode matched filter j is given by

$$P_j(k|k-1) = F_j(k-1)P_{0j}(k-1|k-1)F_j^T(k-1) + \Gamma_j(k-1)Q_j(k-1)\Gamma_j^T(k-1) \tag{9.26}$$

The optimal filtered estimate of the state vector in mode matched filter j is given by

$$\hat{X}_j(k|k) = X_j(k|k-1) + W_j(k)r_j(k) \tag{9.27}$$

where the residual r_j is given by

$$r_j(k) = Z(k) - \hat{Z}_j(k|k-1) \tag{9.28}$$

where $\hat{Z}_j(k|k-1)$ is the measurement prediction given by

$$\hat{Z}_j(k|k-1) = H_j(k)\hat{X}_j(k|k-1) \tag{9.29}$$

The filtered covariance matrix in mode-matched filter j is given by

$$P_j(k|k) = P_j(k|k-1) - W_j(k)S_j(k)W_j^T(k) \tag{9.30}$$

where $S_j(k)$ is the covariance of the residual given by

$$S_j(k) = H_j(k)P_j(k|k-1)H_j^T(k) + R_j(k) \tag{9.31}$$

The Kalman filter gain is given by

$$W_j(k) = P_j(k|k-1)H_j^T(k)S_j^{-1}(k) \tag{9.32}$$

The likelihood function of mode matched filter j is given by

$$\Lambda_j(k) = N[r_j(k); 0, S_j(k)] \tag{9.33}$$

where $N[r_j(k); 0, S_j(k)]$ denotes the multivariate gaussian density function of residual $r_j(k)$ with mean 0 and covariance $S_j(k)$ and $j = 1, 2, \ldots, r$.

The updated mode probability at time k is given by

$$\mu_j(k) = \frac{1}{c}\Lambda_j(k)\sum_{i=1}^{r} p_{ij}\mu_i(k-1) \tag{9.34}$$

Using the normalization factor given by (9.21) in (9.34), we get

$$\mu_j(k) = \frac{1}{c}\Lambda_j(k)\bar{c}_j \tag{9.35}$$

where the likelihood function $\Lambda_j(k)$ is given by (9.33) and c is the normalization constant given by

$$c = \sum_{j=1}^{r}\Lambda_j(k)\bar{c}_j \tag{9.36}$$

9.3.8.3 Combination

Finally, for output only, the latest state estimates and covariances are obtained according to mixture equations

$$\hat{X}(k|k) = \sum_{j=1}^{r}\hat{X}_j(k|k)\mu_j(k) \tag{9.37}$$

$$P(k|k) = \sum_{j=1}^{r}\{P_j(k|k) + B_jB_j^T\}\mu_j(k) \tag{9.38}$$

where B_j is defined as

$$B_j = \hat{X}_j(k|k) - X(k|k) \tag{9.39}$$

9.3.9 Advantages of an IMM Algorithm

Adapted from Ref. 31, the advantages of an IMM algorithm are:

1. It has the ability to estimate the state of a dynamic system with several behavior modes which can switch from one to another.
2. It is a self-adjusting variable bandwidth filter which makes it natural for tracking maneuvering targets.
3. It is the best compromise available currently between complexity and performance.
4. Its computational requirements are nearly linear in the size of the problem (number of models), while its performance is almost the same as that of an algorithm with quadratic complexity.
5. For problems like tracking, the IMM interaction is so effective that IMM algorithm performs almost like the bayesian filter.
6. The IMM requires a far lower computational power than other high-performance algorithms for tracking maneuvering targets.
7. The IMM estimator is one of the most effective and simple schemes for the estimation in hybrid systems and therefore is suitable for multitarget multisensor tracking.
8. The IMM procedure is well established based on a solid theoretical foundation and proved to be appropriate for the maneuvering target tracking problem.
9. It is recursive and modular. It does not require a separate maneuvering following logic.

9.4 SUMMARY

Bar-Shalom–Birmiwal's model is based on the assumption that an aircraft moving with a constant velocity or a constant acceleration motion will eventually lose track if the aircraft deviates from the assumed flight path. Hence a statistical decision test is applied to detect a maneuver. As long as no maneuver is detected, a simpler filter based on a constant velocity model is used for tracking the aircraft. When a maneuver is detected, the tracker is reinitialized using stored data for a higher-order maneuvering model. Then the acceleration filter will run in real time with the arrival of new measurements. Now an end-of-maneuver will monitor the estimated accelerations. Once the acceleration estimate becomes insignificant an end-of-maneuver is declared and the constant velocity filter takes over. Thus the switching between velocity and acceleration filters will take place depending on whether the target is maneuvering or not.

Blom–Bar-Shalom's IMM algorithm uses a bank of parallel filters. The filters, instead of working independently, react with eachother in a probabilistic manner. Due to this interaction the individual filters could adjust their parameters and provide optimum output corresponding to the input. For the purpose of the system output, a weighted average of the individual filter outputs could be taken. The weighting factors are available as part of the filter formulation. There is no need for a separate maneuver detector as in the case of Bar-Shalom–Birmiwal's model.

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10.1 INTRODUCTION

In previous chapters, many algorithms were discussed for tracking a target in a clean environment. Also it was assumed that the return from the target was always received at each scan to update the state estimates of the target (probability of detection is unity). This is only an ideal situation.

In practical applications, this situation may not exist and tracking may have to be performed in an environment of randomly distributed clutter. Clutter refers to radar returns from nearby objects like buildings, water towers, mountains and rains, etc. These false returns are generally random in number, location, and intensity. Even if there is only one target of interest, the number of returns received may be more than one due to clutter and false alarms. This introduces an additional uncertainty regarding the origin of measurements. The problem now is to find out which measurement originated from the target of interest, if it is detected.

Sittler [1] was the pioneer to work out a reasonable method of incorporating measurements of uncertain origin into the existing tracks. This was done before the Kalman filter became popular. Since then several algorithms have been developed based on the Kalman filtering techniques.

The algorithms developed so far for tracking in clutter environment may be classified as non-bayesian and bayesian. The non-bayesian algorithms make decisions to accept or reject possible trajectories based on likelihood functions and then estimate the state, conditioned upon the correctness of these decisions. The resulting state estimates and covariances do not account for the fact that these decisions may be incorrect.

In bayesian methods, the probability that each measurement might be spurious is considered and incorporated into the Kalman filter. A detailed survey of some of these methods is given in Ref. 2, and a comparison of some algorithms is given in Ref. 3.

The problem of tracking a maneuvering target in clutter consists of the following two steps:

1. The association of several detections over a period of time
2. A decision that accepts these detections as having originated from the same target

Prominent among the several algorithms available for tracking a maneuvering target in a clutter environment [1–42] are the probabilistic data association filter (PDAF) developed by Bar-Shalom and Tse [4–8] and the multiple hypotheses tracker (MHT) developed by Reid [5, 9–11].

The PDAF is a suboptimal bayesian algorithm which assumes that there is only one target of interest whose track has been initialized. At each sampling, a validation region is set up. Among the possibly several validated measurements, one can be target originated if the target is detected. The remaining measurements are assumed to be false alarms or residual clutter and are modeled as independent, identically distributed random variables with uniform spatial distributions. This algorithm is discussed in this chapter without details of derivation.

The initial significant work of Blom [13] and his subsequent contribution with Bar-Shalom [14] on interacting multiple model (IMM) estimator boosted the development of algorithms for tracking a maneuvering target in clutter.

In Ref. 15, Blom combined the PDAF with the IMM algorithm in the development of a sophisticated tracking algorithm for ATC surveillance data. In Ref. 16, Houles and Bar-Shalom considered a combination of multisensor PDAF with the IMM algorithm for tracking a highly maneuvering target in clutter with multisensors.

The PDAF in combination with IMM estimator [12, 15, 16] has emerged as the best technique for tracking a maneuvering target in the presence of clutter since it is a recursive algorithm with fixed computation and memory requirements and a minimum of modeling parameters. Also, it achieved an excellent compromise between performance and complexity.

The only disadvantage of the original version of PDAF is that it could not initiate or delete tracks. Recently, Colgrove, Davis, and Ayliffe [7] augmented the PDAF to include initiation and deletion by adding in the association an event corresponding to "unobservable target," which is equivalent to no target. This work motivated the work of Bar-Shalom, Chang, and Blom [12] in developing an algorithm for track formation within the general context of hybrid state (dynamic multiple model) estimation [20]. This algorithm is discussed in this chapter.

10.2 CHANGE OF NOTATIONS

Since the problem is becoming more and more complex, the simple notations used so far are inadequate to describe the complex situations and hence a slight change of notation is introduced henceforth as follows:

The state vector X_k will be denoted as $X(k)$, its filtered estimate $\hat{X}_k$ will be denoted as $\hat{X}(k|k)$, its predicted state estimate $\tilde{X}_k$ as $\hat{X}(k|k-1)$, the filtered covariance $\hat{P}_k$ as $P(k|k)$, and the predicted covariance $\tilde{P}_k$ as $P(k|k-1)$, unless otherwise stated.

10.3 VALIDATION REGION OR GATE

Consider a target track that has already been initiated. The predicted measurement is given by

$$\hat{Z}(k|k-1) = H\hat{X}(k|k-1) \tag{10.1}$$

where $\hat{X}(k|k-1)$ is the predicted estimate of the state vector and H is the observation matrix. The covariance matrix associated with it is given by

$$S(k) = HP(k|k-1)H^T + R(k) \tag{10.2}$$

Then a validation region or gate in the measurement space where the measurement is likely to be found with some high probability [11] is defined as

$$d^2 = \nu^T(k)S^{-1}(k)\nu(k) \tag{10.3}$$

where $\nu(k)$ is the innovation given by

$$\nu(k) = Z(k) - \hat{Z}(k|k-1) \tag{10.4}$$

The validation or gating is then performed by comparing d^2 to a threshold as

$$d^2 \leq G \tag{10.5}$$

Measurements that lie inside the gate are considered valid and those that lie outside are discarded. The parameter G in (10.5) is obtained from the tables of chi-square distribution, since the weighted norm of the innovation (also called the statistical distance) that defines the gate is chi-square distributed with number of degrees of freedom equal to the dimension of the measurement vector.

10.4 PROBABILISTIC DATA ASSOCIATION FILTER (PDAF)

The state of the target is assumed to be described by the dynamic equation

$$X(k+1) = F(k)X(k) + W(k) \tag{10.6}$$

with the measurement given by

$$Z(k) = HX(k) + V(k) \tag{10.7}$$

where W and V are zero mean, mutually independent white gaussian noise sequences with known covariance matrices $Q(k)$ and $R(k)$, respectively. Tracks are assumed to be initiated at $k = 0$.

At each scan, a validation gate given by (10.5), centered around the predicted measurement of the target, is set up to select the measurements to be associated probabilistically to the target.

The simplest approach for tracking a target in a cluttered environment is to select the validated measurement that is closest to the predicted measurement and use it in the tracking filter as if it were the correct one.

The gate or validation region is the region in which the true measurement will appear with a high probability. If more than one measurement is found in the validation region at a given time for a certain target, then any of these validated measurements could have originated from the target. Thus all the measurements in the validation region have to be considered in some way.

In PDAF, the latest set of validated measurements are dealt with. It computes the probabilities of being correct for each validated measurement at the current time, It associates probabilistically all the neighbors to the target of interest. This probabilistic information used in PDAF accounts for the origin uncertainty.

The set of validated measurements at time k is given by

$$Z(k) \triangleq \{Z_i(k)\}_{i=1}^{m(k)} \tag{10.8}$$

and the cumulative set of measurements is given by

$$Z_k = \{Z(\kappa)\}_{\kappa=1}^{k} \tag{10.9}$$

$m(k)$ is the number of measurements in the validation region. The PDAF decomposes the estimation with respect to the origin of each element of the latest set of measurements (10.8).

One cycle of the PDAF [4, 5] is given below without details of derivation.

The best estimate of the target's state is the conditional mean of the state at time k based upon all the observations that with some nonzero probability originated from the target and is given by

$$\hat{X}(k|k) = \sum_{i=0}^{m(k)} \hat{X}_i(k|k)\beta_i(k) \tag{10.10}$$

where $\beta_i(k)$ represents the association probabilities and

$$\sum_{i=0}^{m(k)} \beta_i(k) = 1 \tag{10.11}$$

$\hat{X}_i(k|k)$ is the updated state estimate that the ith validated measurement is correct and is given by

$$\hat{X}_i(k|k) = \hat{X}(k|k-1) + K(k)v_i(k) \tag{10.12}$$

where $i = 1, \ldots, m(k)$ and

$$v_i(k) = Z_i(k) - H\hat{X}(k|k-1) \tag{10.13}$$

$K(k)$ in (10.12) is the gain matrix given by

$$K(k) = P(k|k-1)H^T S^{-1}(k) \tag{10.14}$$

where $S(k)$ is the measurement prediction covariance given by (10.2).

For $i = 0$, i.e., if none of the measurements is correct, then the estimate is

$$\hat{X}_0(k|k) = \hat{X}(k|k-1) \tag{10.15}$$

Combining (10.12) and (10.15) into (10.10) and using (10.11), we get

$$\hat{X}_i(k|k) = \hat{X}(k|k-1) + K(k)\nu(k) \tag{10.16}$$

where

$$\nu(k) = \sum_{i=1}^{m(k)} \beta_i(k)\nu_i(k) \tag{10.17}$$

is known as the combined innovation which uses all the validated measurements.

The error covariance associated with the updated state estimate (10.16) is given by the PDAF [4, 5] as

$$P(k|k) = \beta_0(k)P(k|k-1) + [1-\beta_0(k)]P_k(k|k) + P^*(k) \tag{10.18}$$

where $P_k(k|k)$ is the filtered Kalman covariance matrix that would be computed if a single return were present in the validation region and is given by

$$P_k(k|k) = [I - K(k)H]P(k|k-1) \tag{10.19}$$

and $P^*(k)$ is an increment added to reflect the effect of uncertain correlation and is given by

$$P^*(k) = K(k)\left[\sum_{i=1}^{m(k)} \beta_i(k)\nu_i(k)\nu_i^T(k) - \nu(k)\nu^T(k)\right]K^T(k) \tag{10.20}$$

The predicted state is given by the Kalman filter as

$$\hat{X}(k+1|k) = F(k)\hat{X}(k|k) \tag{10.21}$$

and the covariance of the predicted state is given by

$$P(k+1|k) = F(k)P(k|k)F^T(k) + Q(k) \tag{10.22}$$

The association probabilities for the parametric PDAF with the Poisson clutter model are given by [4, 5]

$$\beta_i(k) = e_i\left(b + \sum_{j=1}^{m(k)} e_j\right)^{-1} \tag{10.23}$$

$$\beta_0(k) = b\left(b + \sum_{j=1}^{m(k)} e_j\right)^{-1} \tag{10.24}$$

where $i = 1, \ldots, m(k)$ and from Refs. 4 and 5, e_i and b are defined as

$$e_i \triangleq \exp[-\tfrac{1}{2} v_i^T(k) S^{-1}(k) v_i(k)] \tag{10.25}$$

$$b \triangleq \lambda[2\pi S(k)]^{1/2} \frac{1 - P_D P_G}{P_D} \tag{10.26}$$

$$= (2\pi/r)^{M/2} \lambda V_k C_M \frac{1 - P_D P_G}{P_D} \tag{10.27}$$

where M is the dimension of the measurement vector and C_M is the volume of the M-dimensional unit hypersphere ($C_1 = 2$, $C_2 = \pi$, $C_3 = 4\pi/3$, and so on [21].

The nonparametric version of the PDAF [5] is the same as above except for replacing λV_k by $m(k)$ in (10.27). For this case, e_i and b may also be defined as [12]

$$e_i \triangleq (P_G)^{-1} N[v_i; 0, S(k)] \tag{10.28}$$

$$b \triangleq m(k)(1 - P_D P_G)[P_D P_G V(k)]^{-1} \tag{10.29}$$

P_D is the probability of detection, $N[v_i; 0, S(k)]$ is the normal probability density function (pdf) with argument v_i, mean zero, and variance $S(k)$; P_G is the probability that the target measurement falls in the validation region, $V(k)$ is the volume of the validation region. For a two-dimensional validation region ("g-sigma gate"), the volume is given by [12]

$$V(k) = g^2 \pi |S(k)|^{1/2} \tag{10.30}$$

$|S(k)|$ is the determinant of $S(k)$.

10.4.1 Advantages of PDAF

1. This is a recursive filter.
2. It has fixed computational requirements, being slightly more complex than a standard Kalman filter.
3. It requires only a minimum of modeling parameters.

The only disadvantage of PDAF is that it cannot initiate or delete tracks. In Ref. 7, Colgrove, Davis, and Ayliffe have augmented the PDAF to include track initiation and deletion by adding in the association an event corresponding to "unobservable target," which is equivalent to "no target."

10.5 BAR-SHALOM–CHANG–BLOM'S MODEL FOR AUTOMATIC TRACK FORMATION

Bar-Shalom–Chang–Blom's model is a recursive track formation algorithm [12] for tracking a maneuvering target in a cluttered environment. This model consists of a combination of the IMM (with two models: "true target" and "no target") with the PDAF to associate the measurements to the tracks that are formed. The PDAF calculates the probabilities of each measurement falling in the validation region that it originated from the target of interest. The nonparametric version of the PDAF [4, 5] is used here. This assumes a known target detection probability, but it does not need the spatial density of the false measurements and hence is suitable for an environment where the false detection rate might change drastically within the surveillance region [12].

10.5.1 Model Formulation

Consider two models, one for observable ("true") target designated as model $t = 2$ and the other for unobservable target ("false target") designated as model $t = 1$.

In model t, let a target originated measurement be detected with probability P_D^t. Then for the observable target, $P_D^2 = P_D$, the target detection probability, and for the unoservable target, $P_D^1 = 0$.

10.5.2 Dynamic Model

Tracking is assumed to be done in the two-dimensional cartesian coordinate system, and the equations of motion of the target are given by

$$X_t(k+1) = F_t X_t(k) + W_t(k) \qquad t = 1, 2 \tag{10.31}$$

where $X_t(k)$, the state vector of the target at time k for model t, is the same for both models and is given by

$$X_t = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} \tag{10.32}$$

F_t is the transition matrix of model t for the sampling period T given by

$$F_t = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad t = 1, 2 \tag{10.33}$$

$W_t(k)$ is the zero-mean white gaussian process noise with known variance

$$E[W_t(k)W_t^T(j)] = Q_t\delta(k, j) \tag{10.34}$$

where

$$Q = \begin{bmatrix} u_t & 0 \\ 0 & u_t \end{bmatrix} \tag{10.35}$$

and

$$u_t = q_t\begin{bmatrix} T^4/4 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix} \qquad t = 1, 2 \tag{10.36}$$

Here, q_t is the variance of the process noise modeling the motion uncertainty (acceleration) in model t.

The state vectors for the two models can be different [15, 16]. In the sequel, the subscript will be dropped for simplicity wherever this does not cause ambiguity.

10.5.3 Measurement Model

The target originated measurements, which occur with probability P_D, are modeled as

$$Z(k) = HX(k) + V(k) \tag{10.37}$$

where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{10.38}$$

$V(k)$ is a zero-mean white gaussian measurement noise with known variance given by

$$E[V_t(k)V_t^T(j)] = R_t\delta(k, j) \tag{10.39}$$

where

$$R_t = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix} \tag{10.40}$$

10.5.4 False Measurements Model

The locations of the false measurements are modeled as uniformly distributed. The number of false measurements is assumed to have a "diffuse prior" (any number of false measurements is equiprobable) distribution [5], which allows a state estimation algorithm that does not require the spatial density of the false measurements (clutter) [12].

10.5.5 Model Transition Probabilities

The observable and unobservable situations are modeled by a Markov chain as follows. Denoting the model in effect during period k by $M(k)$, the following transition (model switching) probabilities are assumed:

$$P\{M(k+1) = \text{unobservable } | M(k) = \text{unobservable}\}$$
$$= p_{11} = 1 - \varepsilon_1 \tag{10.41}$$
$$P\{M(k+1) = \text{observable } | M(k) = \text{unobservable}\}$$
$$= p_{12} = \varepsilon_1 \tag{10.42}$$
$$P\{M(k+1) = \text{unobservable } | M(k) = \text{observable}\}$$
$$= p_{21} = 1 - \varepsilon_2 \tag{10.43}$$
$$P\{M(k+1) = \text{observable } | M(k) = \text{observable}\}$$
$$= p_{22} = 1 - \varepsilon_2 \tag{10.44}$$

That is, transitions between the models are assumed with some low probability.

10.5.6 The IMMPDAF

In Chapter 9, it was assumed that only one measurement, $Z(k)$, is given by the sensor. The extension of IMM to the situation with clutter is obtained as follows:

1. The standard filters in the IMM configuration are replaced by PDAFs of nonparametric version.
2. The calculation of the model probabilities conditioned on the measurements is made using the likelihood function of the PDAF.

Let $M_t(k)$ denote the event that model t is in effect during the kth sampling period. Then $M_s(k-1)$ denotes the event that model s is in effect during period $k-1$.

One cycle of the IMMPDAF algorithm for two models consists of the following four steps.

STEP 1. This is an IMM step [13, 14] described in Chapter 9. Starting with $\hat{X}_s(k-1|k-1)$, the mixed initial condition for the filter matched to model t is computed as

$$\hat{X}_{0t}(k-1|k-1) = \sum_{s=1}^{2} \hat{X}_s(k-1|k-1)\mu_{s|t}(k-1|k-1) \tag{10.45}$$

where

$$\mu_{s|t}(k-1|k-1) = \frac{1}{\bar{c}_t} p_{st}\mu_s(k-1) \tag{10.46}$$

where

$$\bar{c}_t \triangleq \sum_{s=1}^{2} p_{st}\mu_s(k-1) \tag{10.47}$$

and p_{st} is the assumed Markov model-switching probability giving the jump probability from model s at time $k-1$ to model t at time k. These model transition probabilities are assumed known: They are part of the design process, similar to the choice of the model parameters.

The covariance corresponding to (10.45) is

$$P_{0t}(k-1|k-1) = \sum_{s=1}^{2} \mu_{s|t}(k-1|k-1)\{P_s(k-1|k-1) + AA^T\} \tag{10.48}$$

where

$$A = \hat{X}_s(k-1|k-1) - \hat{X}_{0t}(k-1|k-1) \tag{10.49}$$

The estimate (10.45) and covariance (10.48) are used as input to the filter matched to model t to yield $\hat{X}_t(k|k)$ and $P_t(k|k)$.

STEP 2. This is a PDA step [4, 5] discussed in Section 10.4. In the presence of clutter (for a nonparametric PDAF), the likelihood function is the joint probability density function of the innovation, written as

$$\Lambda_t(k) = \left[b(k) + \sum_{j=1}^{m(k)} e_j(k) \right] \cdot \frac{P_D P_G}{m(k)} \cdot V(k)^{-m(k)+1} \tag{10.50}$$

STEP 3. This is a multiple model PDA step [5, 22]. The model probabilities are updated as follows:

$$\mu_t(k) = \frac{1}{c}\Lambda_t(k)\bar{c}_t \tag{10.51}$$

where $\bar{c}_t$ is the expression from (10.47) and $\Lambda_t(k)$ is given in (10.50).

STEP 4 (for output only). The model conditioned estimates and covariances are combined according to the following equations:

$$\hat{X}(k|k) = \sum_{t=1}^{2} \hat{X}_t(k|k)\mu_t(k) \tag{10.52}$$

$$P(k|k) = \sum_{t=1}^{2} \mu_t(k)\{P_t(k|k) + BB^T\} \tag{10.53}$$

with

$$B = \hat{X}_t(k|k) - \hat{X}(k|k) \tag{10.54}$$

10.5.7 Automatic Track Formation Algorithm

The automatic track formation algorithm implemented in Ref. 17 is described in Ref. 12. The tracking operation is done in the two-dimensional cartesian coordinate system. The algorithm is briefly given below:

1. A tentative track is initiated for every detection in the first scan.
2. The velocity of the target along coordinate i is assumed to lie in the interval $[-V_{i\max}, V_{i\max}]$, $i = 1, 2$. A rectangular gate is chosen with its area given by

   $$V(2) = [2(V_{1\max}T + 2\sqrt{R_{11}})][2(V_{2\max}T + 2\sqrt{R_{22}})] \tag{10.55}$$

   Each measurement in the gate yields an initiating pair from which a preliminary track is formed, and the state estimate is initialized based on the first two measurements.
3. From the third scan, the two-model PDAF is run on each preliminary track. A Markov chain transition matrix is assumed between the two models as given in (10.41) to (10.44). Each model is assumed to have initial probability 0.5.
4. The true target probability (TTP) of eack track is computed, and if it falls below a certain threshold, the track is discarded.
5. A test of similarity is done according to the track-to-track association technique [5] to eliminate redundant tracks.

10.5.8 Advantages of Bar-Shalom–Chang–Blom's Model

1. It is a recursive algorithm.
2. It has fixed computational and memory requirements.
3. It can initiate tracks, maintain tracks in the presence of maneuvers, and terminate tracks if warranted.

4. It is useful for situations of low signal-to-noise ratios where the detection threshold has to be set low to detect the targets, this leading to a high rate of false alarms for which logic-based techniques are not adequate.
5. The algorithm yields model probabilities, which provide the true target probability for each track under consideration.
6. The algorithm can assess its own reliability (track loss) and hence may be called "intelligent tracker."

10.6 SUMMARY

Tracking of a maneuvering target in clutter is discussed in this chapter. A change of notations is introduced in Section 10.2. Even if there is only one target of interest, the number of returns received may be more than one due to clutter of false alarms. This introduces an additional uncertainty regarding the origin of measurements. The problem now is to find out which measurement originated from the target of interest, if it is detected. For this a gate or validation region is set up around the predicted measurement at each scan. The gate or validation region is the region in which the true measurement will appear with a high probability. Measurements that lie inside the gate are considered as valid, and those that lie outside the gate are discarded. This is discussed in Section 10.3. If more than one measurement is found in the validation region at a given time for a certain target, then any of these validated measurements could have originated from the target. Thus, all the measurements in the validation region must be considered in some way.

The probabilistic data association filter (PDAF) is a suboptimal bayesian algorithm which assumes that there is only one target of interest whose track has been initialized. In PDAF, the latest set of validated measurements are dealt with. It computes the probabilities of being correct for each validated measurement at the current time. It associates probabilistically all the neighbors to the target of interest. This probabilistic information used in PDAF accounts for the origin uncertainty. This is discussed in Section 10.4. PDAF is a recursive filter with fixed computational requirements and a minimum of modeling parameters. The only disadvantage of the original version of PDAF is that it cannot initiate or delete tracks. Recently, Colgrove, Davis, and Ayliffe [7] augmented the PDAF to include track initiation and deletion by adding in the association an event corresponding to "unobservable target" which can represent either a true target outside the sensor coverage or an erroneously hypothesized target which is equivalent to "no target." This technique enabled PDAF to initiate or delete tracks.

Bar-Shalom–Chang–Blom's model [12] is a recursive track formation algorithm. It consists of a combination of IMM (with two models; "true target" and "no target") with the PDAF to associate the measurements to the tracks that are formed. The PDAF calculates the probabilities of each measurement falling in the validation region that it originated from the target of interest. The nonparametric version of the PDAF is used in this model. This assumes a known target detection probability, but it does not need the spatial density of the false measurements, which makes it suitable for an environment where the false detection rate might change drastically within the surveillance region.

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11.1 INTRODUCTION

Besides the measurement inaccuracies and inadequacies of maneuver models to represent the true target characteristics, the actual tracking problem is much more complicated because of the presence of radar clutter plots or false reports, missing reports due to probability of detection being less than unity, presence of several targets (multiple targets), and also unknown targets requiring track initiation. This introduces an additional uncertainty regarding the origin of measurements, i.e., whether a measurement has originated from a target of interest or not. The number of targets present may also be unknown. This multitarget tracking problem has evoked great interest in recent years because of its application in both military and civilian areas such as ballistic missile defense, air defense, ocean surveillance, battlefield surveillance, air traffic control, etc. Several multitarget tracking algorithms have been developed [1–19] and recent books [6–8, 17–19] discuss the application of these algorithms.

In this chapter, the joint probabilistic data association filter and Reid's multiple hypotheses tracker are briefly mentioned.

11.2 JOINT PROBABILISTIC DATA ASSOCIATION FILTER (JPDAF)

The JPDAF [1–8, 17] is identical to PDAF except that association probabilities are now computed using all observations and all tracks. The state estimates, gain, and covariance of JPDAF are computed using (10.14) to (10.20). The probability computations of (10.23) and (10.24) are now extended to include multiple tracks.

11.3 REID'S MULTIPLE HYPOTHESES TRACKER

Reid's algorithm [9] deals with tracking multiple targets in a cluttered environment. This algorithm is capable of initiating tracks, accounting for false reports due to clutter or missing reports due to probability of detection being less than unity and also processing sets of dependent reports. In this algorithm, as each measurement is received, probabilities are calculated for the data association hypotheses that the measurement came from the previously known targets in a target file, or that the measurement came from a new target requiring track initiation, or that the measurement is false due to a clutter plot. Estimation of target states is made for each such data association hypothesis using a Kalman filter. When more measurements are received, the probabilities of joint hypotheses are calculated recursively using all available information such as density of unknown targets, density of false targets, the probability of detection and the location uncertainty. The number of hypotheses is kept reasonably small by eliminating the unlikely hypotheses and also combining the hypotheses with similar target estimates. The unlikely hypotheses are eliminated if their probabilities are below a specified threshold. Computational requirements are minimized by dividing the entire set of targets and measurements into clusters which are independently solved.

The highlight of the algorithm is to generate a set of data-association hypotheses to account for all possible origins of every measurement. Another interesting feature of this algorithm is the generation of measurement-oriented hypothesis as against the target-oriented hypothesis developed by Bar-Shalom [5]. In the target-oriented hypothesis scheme, every possible measurement is listed for each target, whereas in the measurement-oriented hypothesis scheme, every possible target is listed against each measurement.

Before a new hypothesis is generated, the candidate target must satisfy the following three conditions:

1. If the target is a tentative target, its existence must be implied by the prior hypothesis from which it is branching.
2. Each target is associated with only one measurement.
3. A target is associated with a measurement only if the measurement falls within the validation region of the target.

The validation region is fixed as given in Section 10.3. Each measurement of the data set is associated with a cluster if it falls within a validation region of any target of that cluster for any data association hypothesis of that cluster. A new cluster is formed for each measurement which cannot be associated with any prior cluster. If any measurement is associated with two or more clusters, then those clusters are combined into a super-cluster.

Thus Reid's algorithm [9] incorporates a wide range of capabilities such as a robust data association scheme, track initiation, multiscan correlation, the ability to process data sets with false or missing reports, clustering and recursiveness. The fundamental contribution of this algorithm is the bayesian formulation for determining the probabilities of data to target association hypothesis.

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